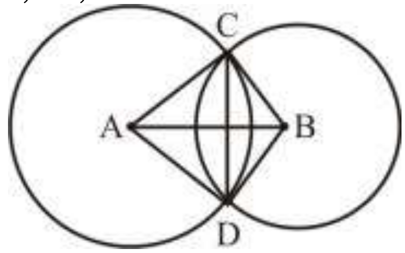
Circles: Exercise 10.6

Q.1 Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

***Sol.* Given:** Let A and B be the two centres of two circles, which intersect each other at points C and D.

To prove: $\angle ACB = \angle ADB$

Construction: Join AC, AD, BD and BC.



Proof: Firstly, in ΔACB and ΔADB,

AC = AD (Radii of the same circle with center A)

BC = BD (Radii of the same circle with center B)

AB = AB (Common side)

So, from SSS criterion of congruence,

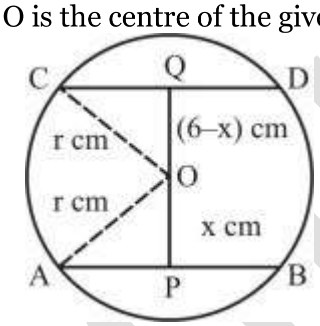
$\Delta ACB \cong \Delta ADB$

$\Rightarrow \angle ACB = \angle ADB$ (By C.P.C.T.)

Hence Proved.

Q.2 Two chords AB and CD of lengths 5 cm and 11 respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.

Sol. Suppose, O is the centre of the given circle and let r be its radius. Now, draw $OP \perp AB$ and $OQ \perp CD$.



Since, given that $OP \perp AB$, $OQ \perp CD$ and $AB \parallel CD$.

So, points P, O and Q are collinear. So PQ = 6 cm.

Let OP = x then, OQ = (6 – x) cm.

Now, join OA and OC.

Then, OA = OC = r.

Since, as we know that perpendicular from the centre to a chord of the circle bisects the chord.

So, AP = PB = 2.5 cm and CQ = QD = 5.5 cm.

Now, in right ΔQAP and ΔOCQ,

$OA^2 = OP^2 + AP^2$

$\Rightarrow r^2 = x^2 + (2.5)^2$ ....(i)

And $OC^2 = OQ^2 + CQ^2$

$r^2 = (6-x)^2 + (5.5)^2$ .... (ii)

From eq. (i) & (ii),

$\Rightarrow x^2 + (2.5)^2 = (6-x)^2 + (5.5)^2$

$\Rightarrow x^2 + 6.25 = 36 - 12x + x^2 + 30.25$

$\Rightarrow 12x = 60$

$\Rightarrow x = 5$

Now, Put $x = 5$ in (i),

$r^2 = 5^2 + (2.5)^2$

$= 25 + 6.25$

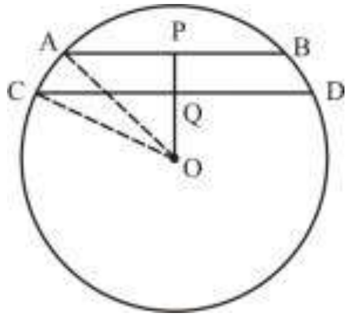
$= 31.25$

$\Rightarrow r = \sqrt{31.25} = 5.6$ (approx.)

Thus, the radius of the circle = 5.6 cm (approx.)

Q.3 The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre.

Sol. **Given:** Let AB = 6 cm and CD = 8 cm be two parallel chords of a circle with centre O and Let r be the radius of the circle.



Construction: construct $OP \perp AB$ and $OQ \perp CD$.

Since, given that AB || CD and $OP \perp AB$, $OQ \perp CD$.

So, points O, Q and P are collinear.

From the figure, OP = 4 cm, and P, Q are mid-points of chords AB and CD respectively.

So, $AP = PB = \frac{1}{2} AB = 3$ cm and, $CQ = QD = \frac{1}{2} CD = 4$ cm

Now, in right angled ΔOAP,

From the Pythagoras theorem,

$OA^2 = OP^2 + AP^2$

$\Rightarrow r^2 = 4^2 + 3^2$

$= 16 + 9$

$= 25$

$\Rightarrow r = 5$

In right angled ΔOCQ,

$OC^2 = OQ^2 + CQ^2$

$\Rightarrow r^2 = OQ^2 + 4^2$

$\Rightarrow 25 = OQ^2 + 16$

$\Rightarrow OQ^2 = 25 - 16$

$OQ^2 = 9$

$\Rightarrow OQ = 3$

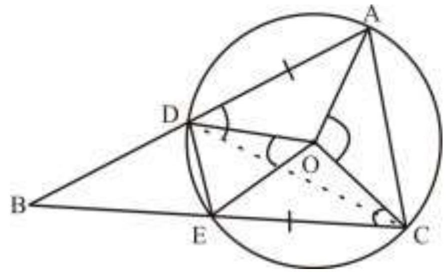
Thus, the distance of chord CD from the centre = 3 cm.

Q.4 Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that ∠ABC is equal to half the difference of the angles subtended by the chords AC and DE at the centre.

Sol. **Given:** Let the vertex B of an angle ABC be located outside the circle and AD = CE.

To prove: According to statement, $\angle ABC = \frac{1}{2}(\angle AOC - \angle DOE)$

Construction: Join DC.



Proof: Since, as we know that an exterior angle of a triangle is equal to the sum of the opposite angles.
So, in ΔBDC,
$\angle ADC = \angle DBC + \angle DCB$............ (i)
And also angle at the centre is twice the angle at a point on the remaining part of circle.

So, $\angle ADC = \frac{1}{2}\angle AOC$ and $\angle DCB = \frac{1}{2}\angle DOE$................ (ii)

From (i) and (ii),

$$\frac{1}{2}\angle AOC = \angle DBC + \frac{1}{2}\angle DOE$$

Since, $\angle DBC = \angle ABC$

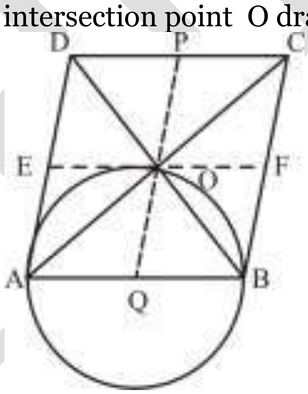
$$\frac{1}{2}\angle AOC = \angle ABC + \frac{1}{2}\angle DOE$$

$\Rightarrow \angle ABC = \frac{1}{2}(\angle AOC - \angle DOE)$........(iii)

Thus, From equation (iii), we can say that $\angle ABC$ is equal to half the difference of angles subtended by the chords AC and DE at the centre.
Hence Proved.

Q.5 Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.

***Sol.* Given:** Let ABCD be a rhombus and AC and BD are its two diagonals which bisect each other at right angles at point O. Now, draw the circle with side AB of a rhombus as diameter.
To prove: A circle drawn on side AB as diameter will pass through O.
Construction: From intersection point O draw PQ || AD and EF || AB.

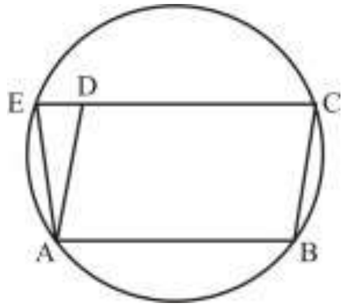


Proof: Since, AB = DC (Sides of rhombus)

$\Rightarrow \quad \frac{1}{2}AB = \frac{1}{2}DC$

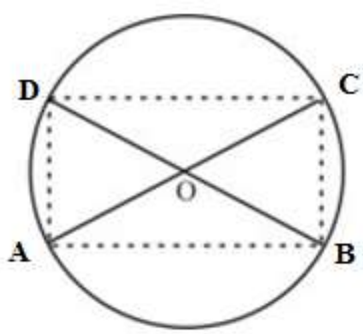
AQ = DP (Since Q and P are mid-points of AB and CD respectively)
Similarly, AE = OQ
$\Rightarrow$ AQ = OQ = QB
$\Rightarrow$ Here, A circle drawn with Q as centre and radius AQ passes through A, O and B. Thus obtained circle is the required circle which passes through A, O and B.

Q.6 ABCD is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that AE = AD.
***Sol.* Given:** ABCD is a parallelogram. A circle passes through points A, B and C intersect CD (produced if necessary) at E on the circumference of the circle.



To prove: AE = AD
Proof: To prove that we need to show ∠AED = ∠ADE. i.e. ΔAED is an isosceles triangle.
Since, from the figure ABCE is a cyclic quadrilateral.
So, ∠AED + ∠ABC = 180°....... (i)
Since, CDE is a straight line.
⇒So, ∠ADE + ∠ADC = 180°.... (ii) (Linear pair angles)
Since, ∠ADC and ∠ABC are opposite angles of a parallelogram
So, ∠ADC = ∠ABC
From (i) & (ii),
∠AED + ∠ABC = ∠ADE + ∠ABC
⇒ ∠AED = ∠ADE
Now, in ΔAED,
∠AED = ∠ADE
⇒So, AD = AE. (Since, opposite sides of equal angles are equal)
Thus, ΔAED is an isosceles triangle.
Hence Proved.

Q.7 AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) ABCD is a rectangle.
***Sol.* Given:** AC and BD are the two chords of a circle with center O. Let they bisect each other at O.
To prove: (i) AC and BD are diameters, (ii) ABCD is a rectangle.
Construction: Join AB, BC, CD and AD.



(i) Proof: Firstly, in ΔAOB and ΔCOD,
OA = OC (Since, O is the mid-point of AC)
∠AOB = ∠COD (Vertically opposite angles)
and, OB = OD (Since, O is the mid-point of BD)
So, from SAS criterion of congruence,
ΔAOB ≅ ΔCOD
⇒ AB = CD (By C.P.C.T)
⇒ Therefore, arc(AB) = arc(CD) .... (i)

Similarly, in ΔAOD and ΔBOC,
arc(AD) = arc(BC)
So, from eq. (i) & (ii),
arc(AB) + arc(AD) = arc(CD) + arc(BC)
⇒ arc(DAB) = arc(BCD)
⇒ So, BD divides the circle into two parts
Thus, BD is a diameter.
Similarly, AB is a diameter.
Hence Proved.

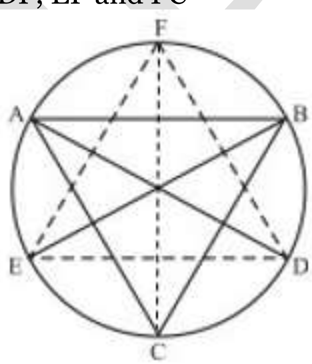
(ii) Proof: Since, already proved $\Delta AOB \cong \Delta COD$
⇒ $\angle OAB$ i.e., $\angle CAB = \angle OBC$ i.e., $\angle BDC$
⇒ So, AB || CD
Again, we have proved that $\Delta AOD \cong \Delta COB$
⇒ So, AD || BC
⇒Therefore, ABCD is a cyclic parallelogram.
⇒ $\angle DAB = \angle DCB$.............. (iii) (Opposite angles of a parallelogram)
And also ACBD is a cyclic quadrilateral.
So, $\angle DAB + \angle DCB = 180^\circ$................ (iv)
From (iii) & (iv),
$\angle DCB + \angle DCB = 180^\circ$
$2\angle DCB = 180^\circ$
$\angle DCB = 90^\circ$ and $\angle DAB = 90^\circ$
$\angle DAB = \angle DCB = 90^\circ$
Thus, ABCD is a rectangle..........Hence Proved.

Q.8 Bisectors of angles A, B and C of a triangle ABC intersect its circumcircle at D, E and F respectively. Prove that the angles of the triangle DEF are $90^\circ - \frac{1}{2}$ A, $90^\circ - \frac{1}{2}$ B, $90^\circ - \frac{1}{2}$ C.

***Sol.* Given:** In a triangle ABC, bisectors of angles A, B and C intersect its circumcircle at D, E and F respectively.

To prove: $\angle D = 90^\circ - \frac{\angle A}{2}$, $\angle E = 90^\circ - \frac{\angle B}{2}$ and $\angle F = 90^\circ - \frac{\angle C}{2}$

Construction: Join ED, DF, EF and FC



Proof: Firstly for $\angle D = \angle EDF$
We can write as $\angle EDF = \angle EDA + \angle ADF$
Since, $\angle EDA$ and $\angle EBA$ are the angles in the same segment of the circle subtended by the same arc AE. Hence $\angle EDA = \angle EBA$
So, $\angle EDF = \angle EBA + \angle ADF$
Similarly, $\angle ADF$ and $\angle FCA$ are the angles in the same segment of the circle subtended by the arc AF.
Hence, $\angle ADF = \angle FCA$
So, $\angle EDF = \angle EBA + \angle FCA$
Since BE is the internal bisector of $\angle B$ and CF is the internal bisector $\angle C$

Therefore, $\angle EDF = \frac{1}{2}\angle B + \frac{1}{2}\angle C$

$\angle D = \frac{\angle B + \angle C}{2}$

Similarly, $\angle E = \frac{\angle C + \angle A}{2}$ and $\angle F = \frac{\angle A + \angle B}{2}$

$\Rightarrow \Rightarrow \angle D = \frac{180^0 - \angle A}{2}$; $\angle E = \frac{180^0 - \angle B}{2}$ and $\angle F = \frac{180^0 - \angle C}{2}$ (Since, $\angle A + \angle B = 180^\circ$)

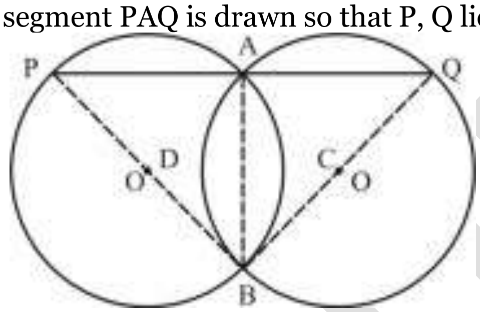
$\Rightarrow \angle D = 90^\circ - \frac{\angle A}{2}$, $\angle E = 90^\circ - \frac{\angle B}{2}$ and $\angle F = 90^\circ - \frac{\angle C}{2}$

$\Rightarrow$ Thus, angles of the ΔDEF: $90^\circ - \frac{1}{2}$ A, $90^\circ - \frac{1}{2}$ B and $90^\circ - \frac{1}{2}$ C.

Hence Proved.

Q.9 Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that BP = BQ.

***Sol.* Given:** Let O and O' be the centre of two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles.



To prove: BP = BQ

Proof: Since, AB is a common chord of these congruent circles.

So, arc (ACB) = arc (ADB)

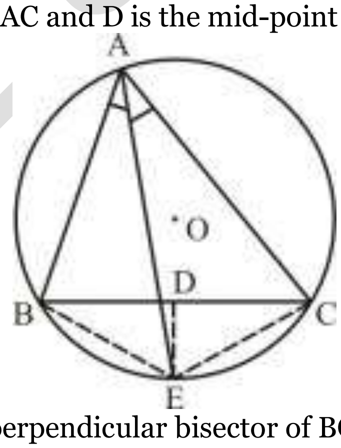
$\angle BPA = \angle BQA$ (Angle made by the same length arc)

$\Rightarrow$Thus, BP = BQ.

Hence Proved.

Q.10 In any triangle ABC, if the angle bisector of ∠A and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the triangle ABC.

Given: In a triangle ABC inscribed in a circle with centre at O, E is a point on the circle such that AE is the internal bisector of ∠BAC and D is the mid-point of side BC.



To Prove: DE is the perpendicular bisector of BC i.e. $\angle BDE = \angle CDE = 90^\circ$.

Construction: Join BE and EC.

Proof: Firstly, in ΔBDE and ΔCDE,

Since, $\angle BAE = \angle CAE$,

Therefore, arc (BE) = arc (CE)

$\Rightarrow$ Chord BE = chord CE

BE = CE

BD = CD (Given)

DE = DE (Common side)

So, from SSS criterion of congruence,

$\Delta BDE \cong \Delta CDE$

$\Rightarrow \angle BDE = \angle CDE$ (By C.P.C.T.)

Also, $\angle BDE + \angle CDE = 180^\circ$ (Linear pair angles)

So, $\angle BDE = \angle CDE = 90^\circ$

Thus, DE is the perpendicular bisector of BC.

Hence Proved.