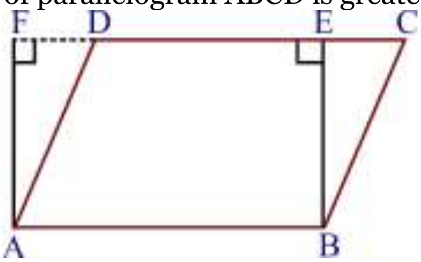
Areas of Parallelogram and Triangles: Exercise 9.4

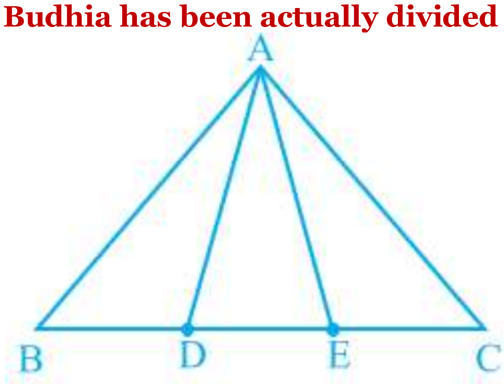
Q.1 Parallelogram ABCD and rectangle ABEF are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

***Sol.* Given:** A parallelogram ABCD and a rectangle ABEF with same base AB and have equal areas.
To prove: Perimeter of parallelogram ABCD is greater than perimeter of rectangle ABEF.

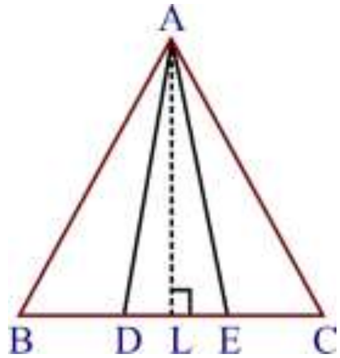


Proof: Since, as we know that opposite sides of a parallelogram and rectangle are equal.
So, AB = DC (Opposite sides of parallelogram ABCD)
and, AB = EF (Opposite sides of rectangle ABEF)
Therefore, DC = EF ... (i)
By adding AB in both sides,
⇒ AB + DC = AB + EF ... (ii)
Since, as we know that all the segments that can be drawn to a given line from a point not lying on it, the perpendicular segment is the shortest distance.
So, BC > BE and AD > AF....(iii)
By adding,
⇒ BC + AD > BE + AF ... (iv)
Now, adding (ii) & (iv),
AB + DC + BC + AD > AB + EF + BE + AF
⇒ AB + BC + CD + DA > AB + BE + EF + FA
⇒ perimeter of parallelogram ABCD > perimeter of rectangle ABEF.
Thus, the perimeter of the parallelogram ABCD is greater than the perimeter of rectangle ABEF.

Q.2 In figure, D and E are two points on BC such that BD = DE = EC. Show that ar (ABD) = ar (ADE) = ar (AEC).
Can you know answer the question that you have left in the 'Introduction' of this chapter, whether the field of Budhia has been actually divided into three parts of equal area?



***Sol.* Given:** In figure, D and E are two points on BC such that BD = DE = EC.
To prove: ar (ABD) = ar (ADE) = ar (AEC).
Construct: Draw AL perpendicular to BC.



Proof: Since, AL ⊥BC. So, AL is the height of ΔABD, ΔADE and, ΔAEC.

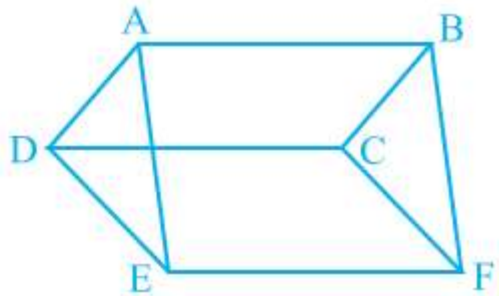
Therefore, ar (ABD) = $\frac{1}{2}$ × BD × AL

ar (ADE) = $\frac{1}{2}$ ×DE × AL

and, ar (AEC) = $\frac{1}{2}$ × EC × AL

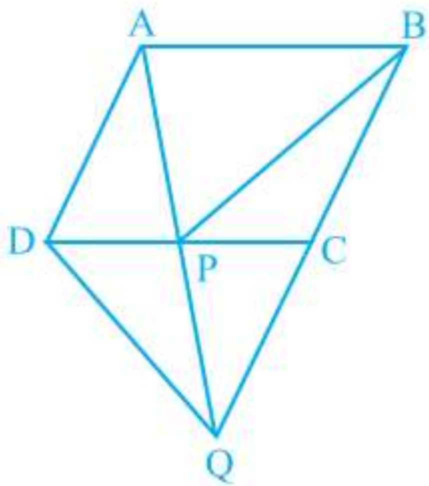
Since, it is given that BD = DE = EC
So, ar (ABD) = ar (ADE) = ar(AEC)...........Hence Proved.
Yes, altitudes of all triangles are same. Budhia will use this result of question for dividing her land in three equal parts.

Q.3 In figure, ABCD, DCFE and ABFE are parallelograms. Show that ar (ADE) = ar (BCF).

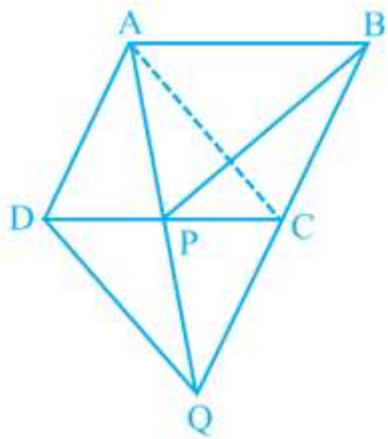


***Sol.* Given:** In figure, ABCD, DCFE and ABFE are parallelograms.
To Prove: ar (ADE) = ar (BCF).
Proof: Since, as we know that opposite sides of a parallelogram are equal.
So, AD = DC (Opposite side of parallelogram ABCD)
DE = CF (Opposite side of parallelogram DCFE)
and AE = BF (Opposite side of parallelogram ABFE)
Now, consider ΔADE and ΔBCF,
AE = BF, AD = BC and DE = CF (Already Proved.)
So, from SSS criterion of congruence,
ΔADE ≅ ΔBCF
⇒ ar(ADE) = ar (BCF)...........Hence Proved.

Q.4 In figure, ABCD is a parallelogram and BC is produced to a point Q such that AD = CQ. If AQ intersect DC at P, show that ar (BPC) = ar (DPQ).



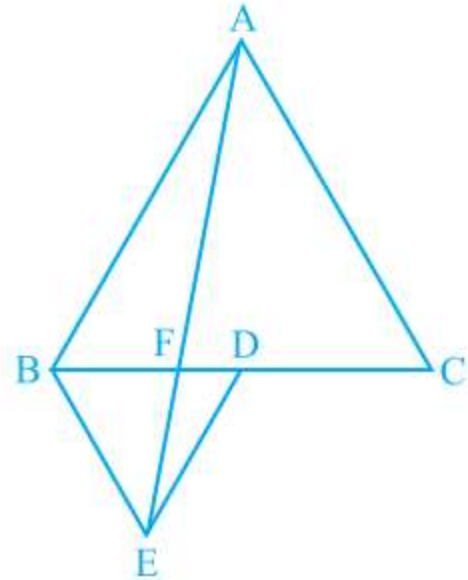
***Sol.* Given:** ABCD is a parallelogram, AD = CQ.
To prove: ar (BPC) = ar (DPQ)
Construct: Join AC.



Proof: From figure, ΔAPC and ΔBPC lie on the same base PC and in between the same parallel lines PC and AB.
So, ar(APC) = ar (BPC) ... (i)
Since AD = CQ and, AD || CQ (Given)
So, in the quadrilateral ADQC, one pair of opposite sides is equal and parallel.
Therefore, quadrilateral ADQC is a parallelogram.
Since diagonals of a parallelogram bisect each other
⇒ AP = PQ and CP = DP (Diagonals of parallelogram ADQC)
Now, in ΔAPC and Δ DPQ,
AP = PQ (Already Proved)
∠APC = ∠DPQ (Vertically opposite angles)
and , PC = PD (Already Proved)
So, from SAS criterion of congruence,
ΔAPC ≅ ΔDPQ
Since congruent triangles have equal area.
⇒So, ar (APC) = ar (DPQ) ... (ii)
Therefore, ar (BPC) = ar (DPQ) (From eq. (i))
Thus, ar (BPC) = ar (DPQ)..............Hence Proved.

Q.5 In figure, ABC and BDE are two equilateral triangles such that D is the mid- point of BC. If AE intersects BC at F, Show that

(i) ar (BDE) = $\frac{1}{4}$ ar (ABC) **(ii) ar (BDE) = $\frac{1}{2}$ ar (BAE)**
(iii) ar (ABC) = 2 ar (BEC) **(iv) ar (BFE) = ar (AFD)**
(v) ar (BFE) = 2 ar (FED) **(vi) ar (FED) = $\frac{1}{8}$ ar (AFC).**

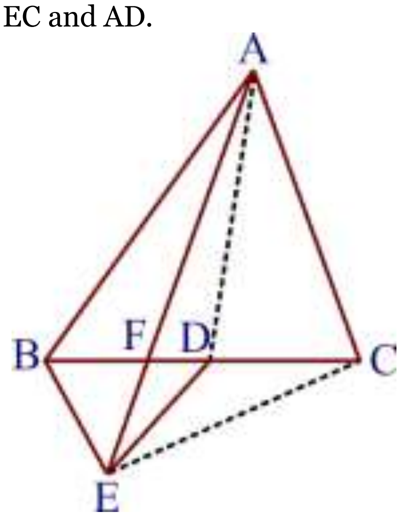


***Sol.* Given:** Two equilateral triangles, ΔABC and BDE such that D is the mid- point of BC. If AE intersects BC at F.

To Prove: (i) ar (BDE) = $\frac{1}{4}$ ar (ABC)

(ii) ar (BDE) = $\frac{1}{2}$ ar (BAE)
(iii) ar (ABC) = 2 ar (BEC)
(iv) ar (BFE) = ar (AFD)
(v) ar (BFE) = 2 ar (FED)
(vi) ar (FED) = $\frac{1}{8}$ ar (AFC).

Construction: Join EC and AD.



Suppose, 'a' is the side of ΔABC. Then, ar(ABC) = $\frac{\sqrt{3}}{4}a^2 = \delta$ (say)

(i) Proof: Since, D is the mid- point of BC. So, BD = $\frac{1}{2}$ BC = $\frac{a}{2}$

So, ar (BDE) = $\frac{\sqrt{3}}{4}\left(\frac{a}{2}\right)^2$

$$= \frac{\sqrt{3}}{16}a^2 = \frac{\delta}{4}$$

⇒ ar (BDE) = $\frac{1}{4}$ ar (ABC)...............Hence Proved.

(ii) Proof: Since, DE is a median of ΔBEC and as we know that each median divides a triangle in two other triangles of equal area.

So, ar (BDE) = $\frac{1}{2}$ ar (BEC) ... (i)

Now, ∠EBC = 60° and ∠BCA = 60° (Angles of equilateral triangles)
⇒So, ∠EBC = ∠BCA
But these angles are alternate angles with respect to line - segments BE and CA and their transversal BC.
Thus, BE || AC.
Now, ΔBEC and ΔBAE are on the same base BE and in between the same parallel lines BE and AC.
So, ar(BEC) = ar(BAE)
Put in equation (i),

Thus, ar (BDE) = $\frac{1}{2}$ ar(BAE)

Hence Proved.

(iii) Proof: Since ED is a median of ΔBEC and as we know that each median divides a triangle in two other triangles of equal area.

Therefore, ar (BDE) = $\frac{1}{2}$ ar (BEC)............ (i)

We have already proved in part (i),

ar (BDE) = $\frac{1}{4}$ ar (ABC)

Now, equating these results,

$\frac{1}{4}$ ar (ABC) = $\frac{1}{2}$ ar (BEC)

⇒ Thus, ar (ABC) = 2 ar (BEC).......Hence Proved.

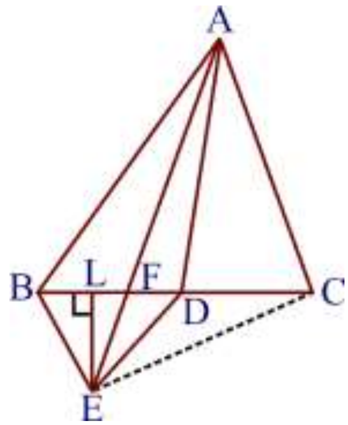
(iv) Proof: Now, ∠ABD = ∠BDE = 60° (Angles of equilateral triangles)
But ∠ABD and ∠BDE are alternate angles with respect to the line- segment BA and DE and their transversal BD.
Therefore, BA || ED.
Now, ΔBDE and ΔAED are on the same base ED and in between the same parallel lines BA and DE.
So, ar (BDE) = ar (AED)
⇒ ar(BDE) – ar(FED) = ar (AED) – ar (FED)
⇒ Thus, ar (BEF) = ar (AFD)............Hence Proved.

(v) Proof: Now in ΔABC,

$AD^2 = AB^2 - BD^2$

$= (a)^2 - (\frac{a}{2})^2$

$= a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$



$\Rightarrow AD = \frac{\sqrt{3}a}{2}$

In Δ BED, $EL^2 = DE^2 - DL^2$

$= (\frac{a}{2})^2 - (\frac{a}{4})^2$

$= \frac{a^2}{4} - \frac{a^2}{16} = \frac{3a^2}{16}$ (Since, EL is median of ΔBED)

$\Rightarrow EL = \frac{\sqrt{3}a}{4}$

So, ar (AFD) = $\frac{1}{2} \times FD \times AD$

$= \frac{1}{2} \times FD \times \frac{\sqrt{3}a}{2}$ ... (i)

and ar(EFD) = $\frac{1}{2} \times FD \times EL$

$= \frac{1}{2} \times FD = \frac{\sqrt{3}a}{4}$ ... (ii)

So, from eq. (i) & (ii), we can write

ar (AFD) = 2 ar (EFD)

Now, combining this result with part (iv).

So, ar (BFE) = ar (AFD) = 2ar (EFD)

Thus, ar (BFE) = 2ar (EFD)............Hence Proved.

(vi) Proof: From proved part (i)

ar (BDE) = $\frac{1}{4}$ ar (ABC)

We can write as

$\Rightarrow$ ar (BEF) + ar (FED) = $\frac{1}{4}$ × 2ar (ADC)

$\Rightarrow$ 2ar (FED) + ar (FED) = $\frac{1}{2}$ (ar(AFC) – ar (AFD) (By using part (v))

$\Rightarrow$ 3ar (FED) = $\frac{1}{2}$ ar (AFC) – $\frac{1}{2}$ × 2 ar (FED)

$\Rightarrow$ 4ar (FED) = $\frac{1}{2}$ ar (AFC)

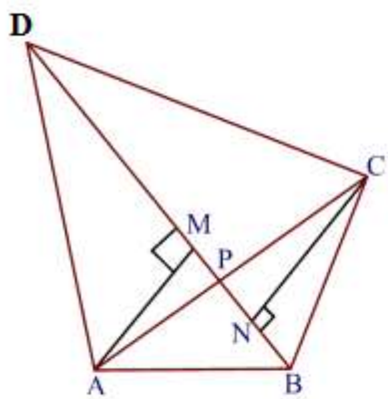
$\Rightarrow$ Thus, ar (FED) = $\frac{1}{8}$ ar (AFC)

Hence Proved.

Q.6 Diagonals AC and BD of a quadrilateral ABCD intersect each other at P. Show that ar (APB) × ar (CPD) = ar (APD) × ar (BPC).

***Sol.* Given:** In a quadrilateral ABCD, Diagonals AC and BD intersect each other at P.

To Prove: ar (APB) × ar (CPD) = ar (APD) × ar (BPC)



Construction: Draw AM ⊥ BD and CN ⊥ BD.

Proof: Now, ar (APB) × ar(CPD) = ($\frac{1}{2}$ × BP × AM) × ($\frac{1}{2}$ × DP × CN)

= $\frac{1}{4}$ × BP × DP × AM × CN ....(i)

and , ar (APD) × ar (BPC) = ($\frac{1}{2}$ × DP × AM) × ($\frac{1}{2}$ × BP × CN)

= $\frac{1}{4}$ × BP × DP × AM × CN.....(ii)

So, from (i) & (ii), we can write,
Thus, ar (APB) × ar (CPD) = ar (APD) × ar (BPC)........Hence Proved.

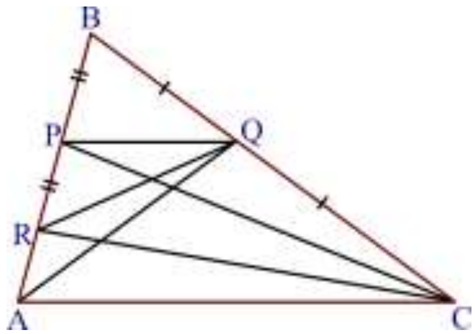
Q.7 P and Q are respectively the mid- points of sides AB and BC of a triangle ABC and R is the mid- point of AP, show that

(i) ar (PRQ) = $\frac{1}{2}$ ar (ARC)

(ii) ar (RQC) = $\frac{3}{8}$ ar (ABC)

(iii) ar (PBQ) = ar (ARC)

***Sol.* Given:** In Δ ABC, P and Q are respectively the mid- points of sides AB and BC and R is the mid- point of AP.



To prove: (i) ar (PRQ) = ar (ARC)

(ii) ar (RQC) = $\frac{1}{2}$ ar (ABC)

(iii) ar (PBQ) = $\frac{3}{8}$ ar (ARC)

Construction: Join AQ and PC.

(i) Proof: Since, QR is a median of Δ APQ and it divides the triangle into two triangles of equal area.

So, ar (PQR) = $\frac{1}{2}$ ar (APQ)

Since, QP is a median of ΔABQ. So, ar (APQ) = $\frac{1}{2}$ ar (ABQ)

Put in above,

$$\text{ar (PQR)} = \frac{1}{2} \times \frac{1}{2} \text{ ar (ABQ)}$$

$$= \frac{1}{4} \text{ ar (ABQ)}$$

Similarly,

$$= \frac{1}{4} \times \frac{1}{2} \text{ ar (ABC) (Since, AQ is a median of ΔABC)}$$

$$= \frac{1}{8} \text{ ar (ABC) ... (i)}$$

Again, ar (ARC) = $\frac{1}{2}$ ar (APC) (Since, CR is a median of ΔAPC)

$$= \frac{1}{2} \times \frac{1}{2} \text{ ar (ABC) (Since, CP is a median of Δ ABC]}$$

$$= \frac{1}{4} \text{ ar (ABC) ... (ii)}$$

So, from eq. (i) & (ii),

ar (PQR) = $\frac{1}{8}$ ar (ABC) = $\frac{1}{2} \times \frac{1}{4}$ ar (ABC)

$= \frac{1}{2}$ ar (ARC).

Hence Proved.

(ii) Proof: From the figure, ar (RQC) = ar(RQA) + ar (AQC) – ar (ARC) ... (iii)
Since RQ is a median of ΔPQA,

So, ar (Δ RQA) = $\frac{1}{2}$ ar (PQA)

$= \frac{1}{2} \times \frac{1}{2}$ ar (AQB) (Since PQ is a median of ΔAQB)

$= \frac{1}{4}$ ar (AQB)

$= \frac{1}{4} \times \frac{1}{2}$ ar(ABC) (Since, AQ is a median of ΔABC)

$= \frac{1}{8}$ ar (ABC) ... (iv)

Now, ar (AQC) = $\frac{1}{2}$ ar (ABC) ... (v) (Since, AQ is a median of ΔABC)

And ar (Δ ARC) = $\frac{1}{2}$ ar (APC)(Since, CR is a median of ΔAPC)

$= \frac{1}{2} \times \frac{1}{2}$ ar(ABC) [Since CP is a median of Δ ABC]

$= \frac{1}{4}$ ar (ABC) ... (vi)

So, from (iii), (iv) , (v) and (vi) we have

ar (RQC) = $\frac{1}{8}$ ar (ABC) + $\frac{1}{2}$ ar (ABC) – $\frac{1}{4}$ ar (ABC)

$= (\frac{1}{8} + \frac{1}{2} - \frac{1}{4})$ ar (ABC)

$= \frac{3}{8}$ ar (ABC)

Thus, ar (RQC) = $\frac{3}{8}$ ar (ABC)............Hence Proved.

(iii) Proof: Since, PQ is a median of ΔABQ

ar (PBQ) = $\frac{1}{2}$ ar (ABQ)

Since, AQ is a median of ΔABC

$$= \frac{1}{2} \times \frac{1}{2} \text{ar(ABC)}$$

$$= \frac{1}{4} \text{ ar (ABC)}$$

By using eq. (vi),

ar (PBQ) = ar(ARC)................Hence Proved.

Q.8 In figure, ABC is a right triangle right angled at A. BCED, ACFG and ABMN are squares on the sides BC, CA and AB respectively. Line segment AX ⊥ DE meets BC at Y. Show that :

(i) ΔMBC ≅ ΔABD

(ii) ar (BYXD) = 2 ar (MBC)

(iii) ar (BYXD) = ar (ABMN)

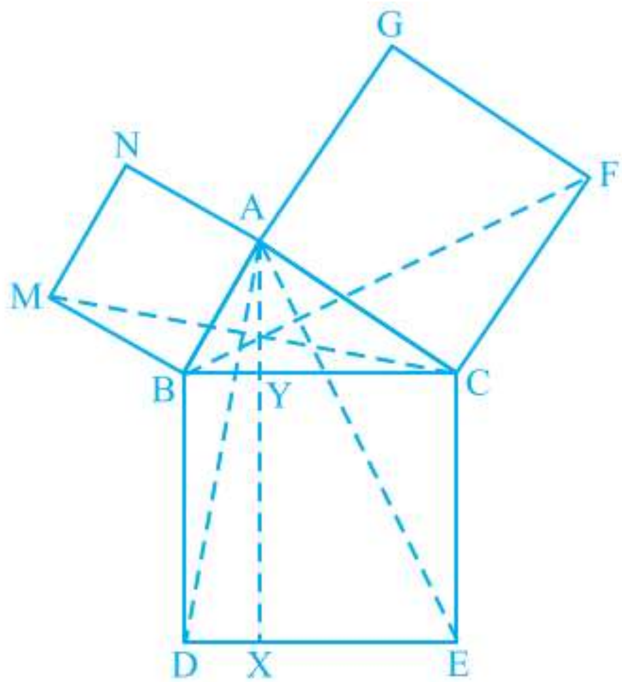
(iv) ΔFCB≅ΔACE

(v) ar (CYXE) = 2 ar (FCB)

(vi) ar (CYXE) = ar (ACFG)

(vii) ar (BCED) = ar (ABMN) + ar(ACFG)

Note : Result (vii) is the famous theorem of Pythagoras. You shall learn a simpler proof of this theorem in class X.



***Sol.* Given:** In right angled triangle ABC right angled at A. BCED, ACFG and ABMN are squares on the sides BC, CA and AB respectively.

To Prove: (i) ΔMBC ≅ ΔABD

(ii) ar (BYXD) = 2 ar (MBC)

(iii) ar (BYXD) = ar (ABMN)

(iv) ΔFCB≅ΔACE

(v) ar (CYXE) = 2 ar (FCB)

(vi) ar (CYXE) = ar (ACFG)

(vii) ar (BCED) = ar (ABMN) + ar(ACFG)

(i) Proof: Firstly, in ΔMBC and ΔABD,

BC = BD (Since, sides of the square BCED)

MB = AB (Since, sides of the square ABMN)

Since, ∠MBC = ∠MBA + ∠ABC (∠MBA = 90^o, Angle of square ABMN)

∠MBC = 90^o + ∠ABC........ (i)

And ∠ABD = ∠DBC + ∠ABC (∠DBC = 90^o, Angle of square BCED)

$\angle ABD = 90^o + \angle ABC$....... (ii)

So, from eq. (i) & (ii), we can write

$\angle MBC = \angle ABD$ (Since, Each = $90^o + \angle ABC$)

So, from SAS criterion of congruence,

$\Delta MBC \cong \Delta ABD$

Hence Proved.

(ii) Proof: Since, from the figure**,** ΔABD and square BYXD are on the same base BD and in between the same parallel lines BD and AX.

So, ar(ABD) = $\frac{1}{2}$ ar (BYXD)

And we have already proved that in part-(i) $\Delta MBC \cong \Delta ABD$

$\Rightarrow$ So, ar(MBC) = ar (ABD)

Therefore, ar(MBC) = ar (ABD) = $\frac{1}{2}$ ar (BYXD)

$\Rightarrow$ Thus, ar(BYXD) = 2 ar (MBC).

Hence Proved.

(iii) Proof: Again from the figure, square ABMN and ΔMBC are on the same base MB and in between same parallel lines MB and NC.

So, ar(MBC) = $\frac{1}{2}$ ar (ABMN)

$\Rightarrow$ ar(ABMN) = 2 ar (MBC)

Since in part-(ii) we have already proved that ar(BYXD) = 2 ar (MBC).

So, ar(ABMN) = 2 ar (MBC)

(iv) Proof: Now, In ΔACE and ΔBCF,

CE = BC (Sides of the square BCED)

AC = CF (Sides of the square ACFG)

Since, $\angle ACE = \angle BCE + \angle BCA$ ($\angle BCE = 90^o$, angle of square BCDE)

$\angle ACE = 90^o + \angle BCA$ .... (i)

And $\angle BCF = \angle ACF + \angle BCA$ ($\angle ACF = 90^o$, angle of square ACFG)

$\angle BCF = 90^o + \angle BCA$........... (ii)

From eq. (i) & (ii),

$\angle ACE = \angle BCF$

So, from SAS criterion of congruence,

Thus, $\Delta ACE \cong \Delta BCF$..........Hence Proved.

(v) Proof: Since, from the figure, ΔACE and square CYXE are on same base CE and in between same parallel lines CE and AX.

So, ar(ACE) = $\frac{1}{2}$ ar (CYXE)

And we have already proved in part (iv) that $\Delta ACE \cong \Delta BCF$

$\Rightarrow$So, ar(FCB) = $\frac{1}{2}$ ar (CYXE)

$\Rightarrow$Thus, ar(CYXE) =2 ar (FCB).............Hence Proved.

(vi) Proof: From the figure, square ACFG and ΔBCF are on same base CF and in between same parallel lines CF and BAG.

So, ar(BCF) = $\frac{1}{2}$ ar(ACFG)

And we have already proved in part- (v) that ar(CYXE) =2 ar (FCB)

⇒So, $\frac{1}{2}$ ar(CYXE) = $\frac{1}{2}$ ar (ACFG)

⇒ Thus, ar(CYXE) = ar (ACFG)………….Hence Proved.

(vii) Proof: We have already proved in part (iii) and (vi) that
ar (BYXD) = ar (ABMN) and ar (CYXE) = ar (ACFG)
By adding both,
ar (BYXD) + ar(CYXE) = ar (ABMN) + ar(ACFG)
Thus, ar (BCED) = ar (ABMN) + ar(ACFG)……………Hence Proved.