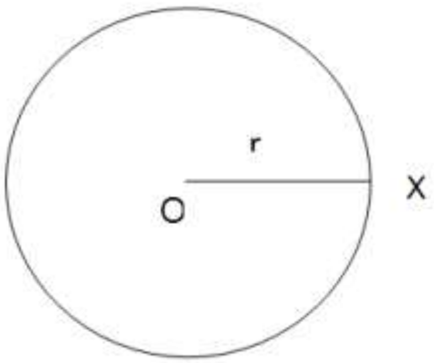
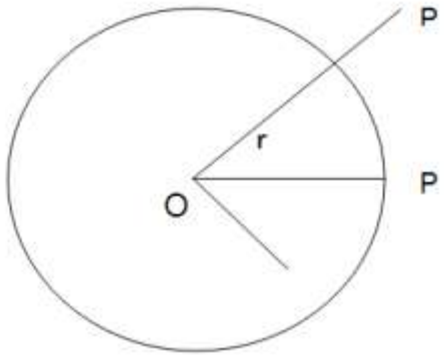
Circles

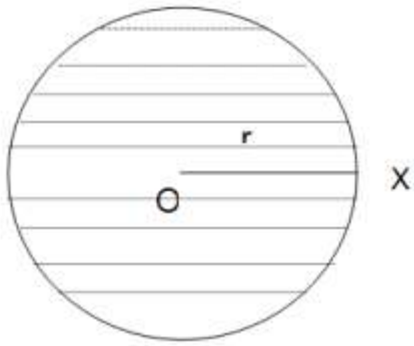
(1) A circle is the collection of those points in a plane that are at a given constant distance from a fixed-point in the plane. The fixed point is called the centre and the given constant distance is called the radius of the circle.
A Circle with centre O and radius r usually denoted by C(O, r). Thus, in set theoretic notations, we write C(O, r) = {X : OX = r}



(2) A point P lies inside or on or outside the circle C(O, r) according as OP < r or OP = r or OP > r.



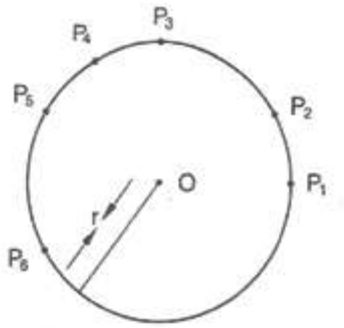
(3) The collection of all points lying inside and on the circle C(O, r) is called a circular disc with centre O and radius r.
The set of all points lying inside and on the circle is called a Circular Disc. It is also known as the circular region.



(4) Circles having the same centre and different radii are said to be concentric circles.
When two or more circles have the same centre but have different radii, they are called as concentric circles, that is, circles with common centre.

(5) A continuous piece of a circle is called an arc of the circle.

For Example: Consider circle C (O, r). Let P1, P2, P3, P4, P5, P6 be point on the circle. Then, the pieces P1,P2,P3,P4,P5,P6,P1,P2 etc. are all arcs of the circle C(O, r).



(6) Prove that If two arcs of circle are congruent, then corresponding chords are equal.

Given: Arc PQ of a Circle C(O, r) and arc RS of another circle C(O′, r) such that PQ ≅ RS

To Prove: PQ = RS

Construction: Draw Line segment OP, OQ, O′R and O′S.

Proof:

Case-I When arc(PQ) and arc(RS) are minor Arcs

In triangle OPQ and O′RS, We have

OP = OQ = O′R = O′S = r [Equal radii of two circles]

∠POQ = ∠RO′S

arc (PQ) ≅ arc (RS) ⇒ m(arc(PQ)) ≅ m(arc(RS)) ⇒ ∠POQ = ∠RO′S

So by SAS Criterion of congruence, we have

ΔPOQ ≅ ΔRO′S

⇒ PQ = RS

Case-II When arc (PQ) and arc (RS) are major arcs.

If arc (PQ), arc (RS) are major arcs, then arc (QP) and arc (SR) are Minor arcs.

So arc (PQ) ≅ arc (RS)

⇒ arc(QP) ≅ arc(SR)

⇒ QP = SR

⇒ PQ = RS

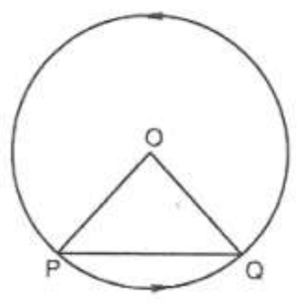
Hence, PQ ≅ RS ⇒ PQ = RS

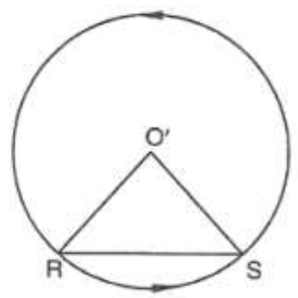
(7) Prove that If two chords of a circle are equal, then their corresponding arcs are congruent.

Given: Equal chords, PQ of a circle C(O, r) and RS of congruent circle C(O′, r)

To Prove: arc (PQ) ≅ arc (RS), where both arc (PQ) and arc(RS) are minor, major or semi-circular arcs.

Construction: If PQ, RS are not diameters, draw line segments OP, OQ, O′R and O′S.



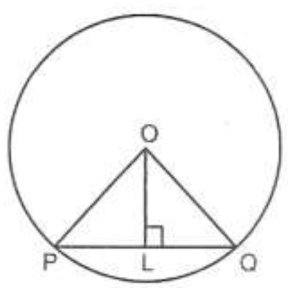


Proof:
Case I: when arc (PQ) and arc (RS) are diameters
In this case, PQ and RS are semi-circle of equal radii, hence they are congruent.

Case II: When arc (PQ) and arc (RS) are Minor arcs.
In triangles POQ and RO′S, we have
$PQ = RS$
$OP = O'R = r$ and $OQ = O'S = r$
So by SSS-criterion of congruence, we have
$\Delta POQ \cong \Delta RO'S$
$\Rightarrow \quad \angle POQ = \angle RO'S$
$\Rightarrow \quad m\ (arc\ (PQ)) = m\ (arc\ (RS))$
$\Rightarrow \quad arc\ (PQ) \cong arc\ (RS)$

Case III: When arc (PQ) and arc (RS) are major arcs
In this case, arc (QP) And arc (SR) will be minor arcs.
$PQ = RS$
$\Rightarrow \quad QP = SR$
$\Rightarrow \quad m(arc(QP)) = m(arc(SR))$
$\Rightarrow \quad 360° - m(arc(PQ)) - 360° - m(arc(RS))$
$\Rightarrow \quad m(arc(PQ)) - m(arc(RS))$
$\Rightarrow \quad arc\ (PQ) \cong arc\ (RS)$
Hence, in all the three cases, we have $arc(PQ) \cong arc(RS)$

(8) Prove that The perpendicular from the centre of a circle to a chord bisects the chord.
Given: A Chord PQ of a circle C(O, r) and perpendicular OL to the chord PQ.
To Prove: $LP = LQ$
Construction: Join OP and OQ



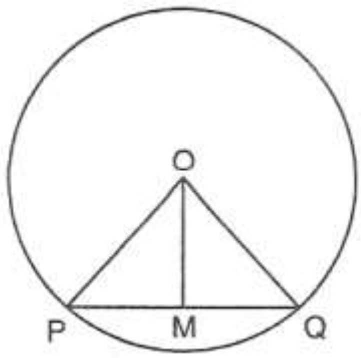
Proof: In Triangles PLO and QLO, we have
$OP = OQ = r$ [Radii of the same circle]
$OL = OL$ [Common]
And, $\angle OLP = \angle OLQ$ [Each equal to 90°]
So, by RHS-criterion of congruence, we have
$\Delta PLO \cong \Delta QLO$
$\Rightarrow \quad PL = LQ$

(9) Prove that The line segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

Given: A Chord PQ OF a circle C(O, r) with mid-point M.

To Prove: $OM \perp PQ$

Construction: Join OP and OQ



Proof: In triangles OPM and OQM, we have

$OP = OQ$ [Radii of the same circle]

$PM = MQ$ [M is mid-point of PQ]

$OM = OM$

So, by SSC - criterion of congruence, we have

$\Delta OPM \cong \Delta OQM$

$\Rightarrow \quad \angle OMP = \angle OMQ$

But , $\angle OMP + \angle OMQ = 180°$ [Linear pair]

$\Rightarrow \quad \angle OMP + \angle OMP = 180°$ [$\angle OMP = \angle OMQ$]

$\Rightarrow \quad 2\angle OMP = 180°$

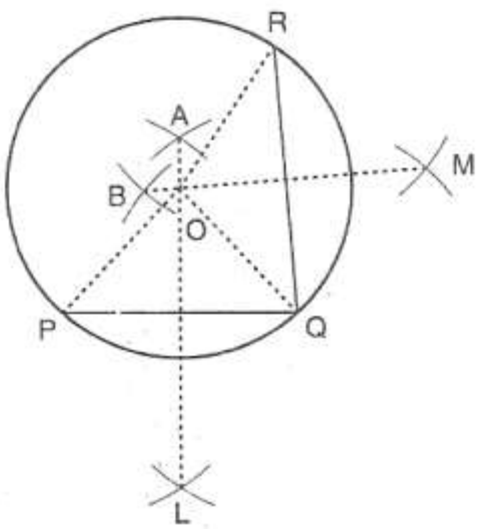
$\Rightarrow \quad \angle OMP = 90°$

(10) Prove that There is one and only circle passing through three given points.

Given: Three non-collinear points P, Q and R.

To Prove: There is one and only one circle passing through P, Q and R.

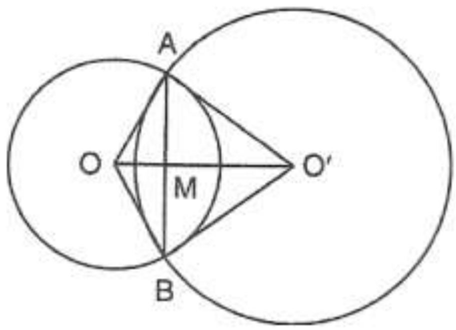
Construction: Join PQ and QR. Draw perpendicular bisectors AL and BM of PQ and RQ respectively. Since P, Q and R. are not collinear. Therefore, the perpendicular bisectors AL and BM are not parallel. Let AL and BM intersect at O. Join OP, OQ and OR.



Proof: Since O lies on the perpendicular bisector of PQ.

Therefore,
$OP = OQ$
Again, O Lies on the perpendicular bisector of QR.
Therefore,
$OQ = OR$
Thus, $OP = OQ = OR = r$ (say)
Taking O as the centre draw a circle of radius s. Clearly, C(O, s) passes through P, Q and R. This proves that there is a circle passing the points P, Q and R.
We shall now prove that this is the only circle passing through P, Q and R.
If possible, let there be another circle with centre O′ and radius r, passing through the points P, Q and R. Then, O′ will lie on the perpendicular bisectors AL of PQ and BM of QR.
Since two lines cannot intersect at more than one point, so O′ must coincide with O. Since $OP = r$, $O'P = s$ and O and O′ coincide, we must have $r = s$
$\Rightarrow \quad C(O, r) = C(O', s)$
Hence, there is one and only one circle passing through three non-collinear points P, Q and R.

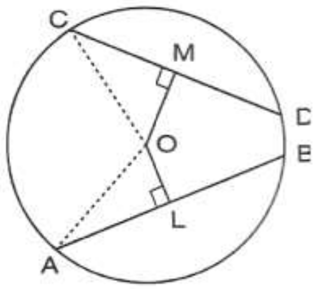
(11) Prove that If two circles intersect in two points, then the through the centre is perpendicular to the common chord.
Given: Two circles C(O, r) and C(O′,s) intersecting at points A and B.
To Prove: OO′ is perpendicular bisector of AB.
Construction: Draw line segments
OA, OB, O′A and O′B



Proof: In triangles OAO′ and OBO′, we have
$OA = OB = r$
$O'A = O'B = s$
And, $OO' = OO'$
So, by SSS -criterion of congruence, we have
$\Delta OAO \cong \Delta OBO'$
$\Rightarrow \quad \angle AOO' = \angle BOO'$
$\Rightarrow \quad \angle AOM = \angle BOM \qquad [\angle AOO' = \angle AOM \text{ and } \angle BOM = \angle BOO']$
Let M be the point of intersection of AB and OO′
In triangles AOM and BOM, we have
$OA = OB = r$
$\Rightarrow \quad \angle AOO' = \angle BOO'$
$\Rightarrow \quad \angle AOM = \angle BOM \qquad [\angle AOO' = \angle AOM \text{ and } \angle BOM = \angle BOO']$
Let M be the point of intersection of AB and OO′
In triangles AOM and BOM, we have
$OA = OB = r$
$\angle AOM = \angle BOM$
And $\quad OM = OM$
So, by SAS-criterion of congruence, we have
$\Delta AOM \cong \Delta BOM$

$\Rightarrow$ AM = BM and $\angle AMO = \angle BOM$
But, $\angle AOM + \angle BMO = 180°$
$2\angle AOM = 180°$
$\Rightarrow$ $\angle AOM = 90°$
Thus, $AM = BM$ and $\angle AOM = \angle BMO = 90°$
Hence, OO′ is the perpendicular bisector of AB.

(12) Prove that Equal chords of a circle subtend equal angle at the centre.
Given: Two Chord AB and CD of circle C(O,r) such that AB=CDand OL⊥AB and OM⊥CD
To Prove: Chord AB and CD are equidistant from the centre O i.e OL=OM.
Construction: Join OA and OC.



Proof: Since the perpendicular from the centre of a circle to a chord bisects the chord.
Therefore,

$OL \perp AB \Rightarrow AL = \frac{1}{2} AB$..........(i)

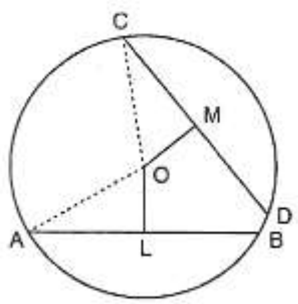
And, $OM \perp CD \Rightarrow CM = \frac{1}{2} CD$..........(ii)

But, AB=CD

$\Rightarrow \frac{1}{2} AB = \frac{1}{2} CD$

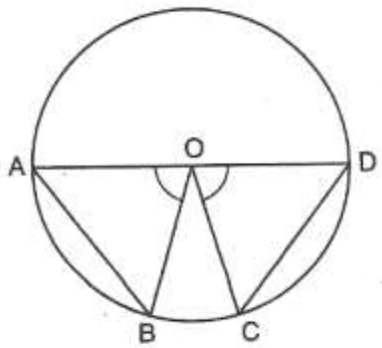
$\Rightarrow$ AL=CM [Using (i) and (ii)]..........(iii)
Now, in right triangles OAL and OCM, we have
OA=OC [Equal to radius of the circle]
AL=CM [From equation (iii)]
And, $\angle ALO = \angle CMO$ [Each equal to 90°]
So by RHS criterion of convergence, we have
$\Delta OAL \cong \Delta OCM$
$\Rightarrow$ OL=OM
Hence, equal chord of a circle are equidistant from the centre.

(13) Prove that Chords of a circle which are equidistant from the centre are equal.
Given: Two Chords AB and CD of a circle C(O,r) which are equidistant from its centre i.e. OL=OM, where OL⊥AB and OM⊥CD.
To Prove: Chords are Equal i.e. AB=CD
Construction: Join OA and OC



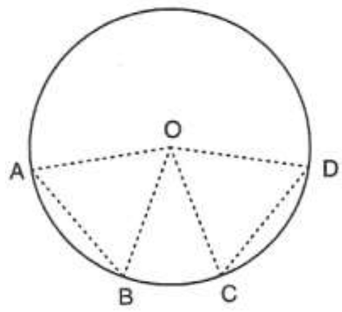
Proof: Since the perpendicular from the centre of a circle to a chord bisects the chord.
Therefore,
OL⊥AB
⇒ $AL = BL$
⇒ $AL = \frac{1}{2} AB$
And, OM ⊥ CD
⇒ $CM = DM$
⇒ $CM = \frac{1}{2} CD$
In triangles OAL and OCM, we have
$OA = OC$ [Each equal to radius of the given Circle]
$\angle OLA = \angle OMC$ [Each equal to 90∘]
And, OL=OM [Given]
So, by RHS, criterion of convergence, we have
$\Delta OAL \cong \Delta OCM$
⇒ $AL = CM$
⇒ $\frac{1}{2} AL = \frac{1}{2} AB$
⇒ $AB = CD$
Hence, the chords of a circle which are equidistant from the centre are equal.

(14) Prove that Equal chords of a circle subtend equal angle at the centre.
Given: A circle C(O,r) and its two equal chords AB and CD.
To Prove: $\angle AOB = \angle COD$



Proof: In triangles AOB and COD, we have
$AB = CD$ [Given]
$OA = OC$ [Each equal to r]
$OB = OD$ [Each equal to r]
So, by SSC-criterion of Congruence, we have
$\Delta AOB \cong \Delta COD$
⇒ $\angle AOB = \angle COD$

(15) Prove that If the angles subtended by two chords of a circle at the centre are equal, the chords are equal.
Given: Two Chord AB and CD of a circle C(O, r) such that ∠AOB=∠COD
To Prove: AB=CD



Proof: In triangles AOB and COD, we have

$OA = OC$ [Each equal to r]

$\angle AOB = \angle COD$ [Given]

$OB = OD$ [Each equal to r]

So, by SAS-criterion of congruence, we have

$\Delta AOB \cong \Delta COD$

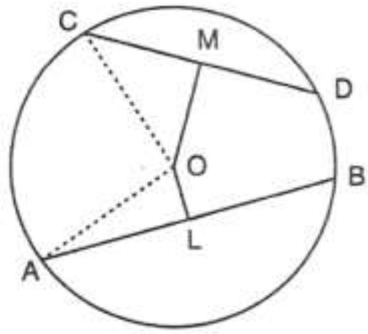
$\Rightarrow \quad AB = CD$

(16) Prove that Of any two chords of a circle, the larger chord is nearer to the centre.

Given: Two Chord AB and CD of a circle with Centre O such that AB>CD

To Prove: Chord AB is nearer to the centre of the circle i.e. OL<OM, where OL and OM are perpendiculars from O to AB and CD respectively

Construction: Join OA and OC.



Proof: Since the perpendicular from the centre of a circle to a chord bisects the chord. Therefore,

$OL \perp AB \Rightarrow AL = 12AB$

And, $OM \perp CD \Rightarrow CM = \frac{1}{2} CD$

In right triangles OAL and OCM, we have

$OA^2 = OL^2 + AL^2$

And, $OC^2 = OM^2 + CM^2$

$\Rightarrow \quad OL^2 + AL^2 = OM^2 + CM^2$.. (i) $[OA = OC \Rightarrow OA^2 = OC^2]$

Now, $AB > CD$

$\Rightarrow \quad \frac{1}{2} AB > \frac{1}{2} CD$

$\Rightarrow \quad AL > CM$

$\Rightarrow \quad AL^2 > CM^2$

$\Rightarrow \quad OL^2 + AL^2 > OL^2 + CM^2$ [Adding OL^2 on both sides]

$\Rightarrow \quad OM^2 + CM^2 > OL^2 + CM^2$ [using equation (i)]

$\Rightarrow \quad OM^2 > OL^2$

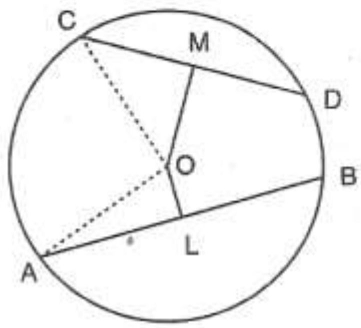
$\Rightarrow \quad OM > OL$

$\Rightarrow \quad OL < OM$

Hence, AB is nearer to the centre than CD.

(17) Prove that Of any two chords of a circle, the chord nearer to the centre is larger.

Given: Two Chord AB and CD of a circle C(O,r) such that OL<OM, where OL and OM are perpendiculars From O on AB and CD respectively.
To Prove: AB>CD
Construction: Join OA and OC.



Proof: Since the perpendicular From the Centre of a circle to a chord bisects the chord.

$AL = \frac{1}{2} AB$ and $CM = \frac{1}{2} CD$

In right triangles OAL and OCM, we have

$OA^2 = OL^2 + AL^2$ and, $OC^2 = OM^2 + CM^2$

$\Rightarrow \quad AL^2 = OA^2 - OL^2$ .... (i)

And, $\quad CM^2 = OC^2 - OM^2$.......(ii)

Now, $\quad OL < OM$

$\Rightarrow \quad OL^2 < OM^2$

$\Rightarrow \quad -OL^2 > -OM^2$

$\Rightarrow \quad OA^2 - OL^2 > OA^2 - OM^2$ [adding OA^2 on both sides]

$\Rightarrow \quad OA^2 - OL^2 > OC^2 - OM^2$ [$OA^2 = OC^2$]

$\Rightarrow \quad AL^2 > CM^2$

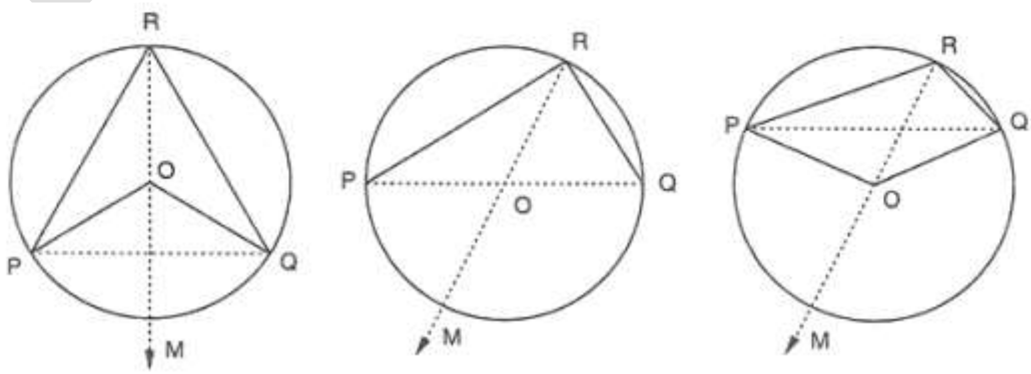
$\Rightarrow \quad AL > CM$

$\Rightarrow \quad 2AL > 2CM$

$\Rightarrow \quad AB > CD$

(18) Prove that The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Given: An arc PQ of a circle C(O, r) and a point R on the remaining part of the circle i.e. arc QP.
To Prove: ∠POQ=2∠PRQ
Construction: join RO and produce it to a point M outside the circle.



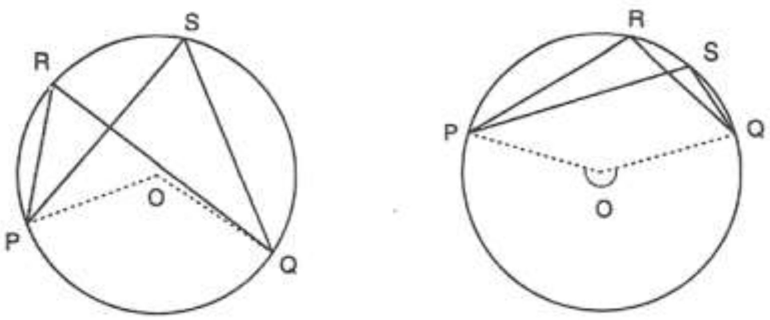
Proof: We shall consider the following three different cases:

Case I: when arc (PQ) is a minor arc.
We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles.
In ΔPOQ, ∠POM is the exterior angle.
∠POM = ∠OPR + ∠ORP
⇒ ∠POM = ∠ORP + ∠ORP   [OP = OR = r, ∠OPR = ∠ORP] ⇒ ∠POM = 2∠ORP…………………(i)
In ΔQOR, ∠QOM is the exterior angle.
∠QOM = ∠OQR + ∠ORQ
⇒ ∠QOM = ∠OQP + ∠ORQ   [OQ = OR = r, ∠ORQ = ∠OQR]
⇒ ∠QOM = 2∠ORQ…………….(ii)
Adding equation (i) and (ii), we get
∠POM + ∠QOM = 2∠ORP + 2∠ORP
⇒ ∠POM + ∠QOM = 2(∠ORP + ∠ORP)
⇒ ∠POM = 2∠PRQ

Case II: when arc (PQ) is a semi-circle
We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles.
In ΔPOQ, we have
∠POM = ∠OPR + ∠ORP
⇒ ∠POM = ∠ORP + ∠ORP   [OP = OR = r, ∠OPR = ∠ORP]
⇒ ∠POM = 2∠ORP…………………(iii)
In ΔQOR, We have
∠QOM = ∠ORQ + ∠OQR
⇒ ∠QOM = ∠ORQ + ∠ORQ   [OQ = OR = r, ∠ORQ = ∠OQR]
⇒ ∠QOM = 2∠ORQ…………….(iv)
Adding equations (iii) and (iv), we get
∠POM + ∠QOM = 2(∠ORP + ∠ORQ)
∠POQ = 2∠PRQ

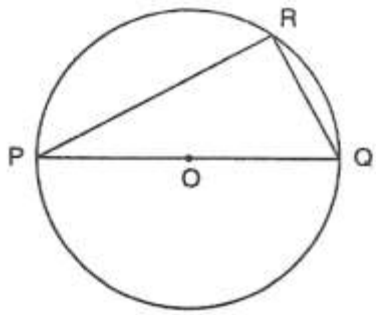
Case III: When arc (PQ) is a major arc.
We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles
In ΔPOR, we have
∠POM = ∠ORP + ∠ORP   [OP = OR = r, ∠OPR = ∠ORP]
⇒ ∠POM = 2∠ORP ……………….(v)
In ΔQOR, we have
∠QOM = ∠ORQ + ∠OQR
⇒ ∠QOM = 2∠ORQ ……………….(vi)
Adding equations (v) and (vi), we get
∠POM + ∠QOM = 2(∠ORP + ∠ORP)
⇒ Reflex ∠POQ = 2∠PRQ

(19)Prove that Angles in the same segment of a circle are equal.
Given: A circle C(O, r), an arc PQ and two angles ∠PRQ and ∠PSQ in the same segment of the circle.
To Prove: ∠PRQ= ∠PSQ
Construction: Join OP and OQ



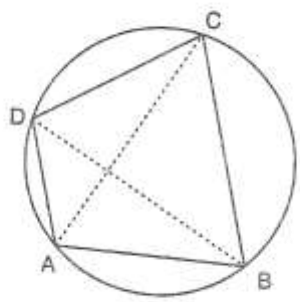
Proof: we know that the angle subtended by an arc at the centre is double the angle subtended by the arc at any point in the remaining part of the circle. So we have
$\angle POQ=2\angle PRQ$ and $\angle POQ=2\angle PSQ$
$\Rightarrow 2\angle PRQ=2\angle PSQ$
$\Rightarrow \angle PRQ=\angle PSQ$
We have
Reflex $\angle POQ=2\angle PRQ$ and $\angle POQ=2\angle PSQ$
$\Rightarrow 2\angle PRQ=2\angle PSQ$
$\Rightarrow \angle PRQ=\angle PSQ$
Thus, in both the cases, we have
$\angle PRQ=\angle PSQ$

(20) Prove that The angle in a semi-circle is a right angle.
Given: PQ is a diameter of a circle C(O, r) and $\angle PRQ$ is an angle in semi-circle.
To Prove: $\angle POQ=90°$



Proof: we know that the angle subtended by an arc of a circle at its centre is twice the angle formed by the same arc at a point on the circle. So, we have
$\angle POQ=\angle PRQ$
$\Rightarrow 180°=2\angle PRQ$ [POQ is a straight line]
$\Rightarrow \angle PRQ=90°$

(21) Prove that The opposite angles of a cyclic quadrilateral are supplementary.
Given: A Cyclic quadrilateral ABCD
To Prove: $\angle A+\angle C=180∘$ and $\angle B+\angle D=180∘$
Construction: Join AC and BD.



Proof: Consider side AB of quadrilateral ABCD as the Chord of the circle. Clearly, $\angle ACB$ and $\angle ADB$ are angles in the same segment determined by chord AB of the Circle.
$\angle ACB = \angle ADB$ …………(i)
Now, consider the side BC of quadrilateral ABCD as the chord of the circle. We find that $\angle BAC$ and $\angle BDC$ are angles in the same segment
$\angle BAC = \angle BDC$ [angles in the same segment are equal]..(ii)

Adding equation (i) and (ii), we get

$\Rightarrow \quad \angle ACB + \angle BAC = \angle ADB + \angle BDC$

$\Rightarrow \quad \angle ACB + \angle BAC = \angle ADC$

$\Rightarrow \quad \angle ABC + \angle ACB + \angle BAC = \angle ABC + \angle ADC$

$\Rightarrow \quad 180° = \angle ABC + \angle ADC$ [sum of angle of triangle is 180°]

$\Rightarrow \quad \angle ABC + \angle ADC = 180°$

$\Rightarrow \quad \angle B + \angle D = 180°$

But, $\angle A + \angle B + \angle C + \angle D = 360°$

$\angle A + \angle C = 360° - (\angle B + \angle D)$

$\Rightarrow \angle A + \angle C = 360° - 180° = 180°$

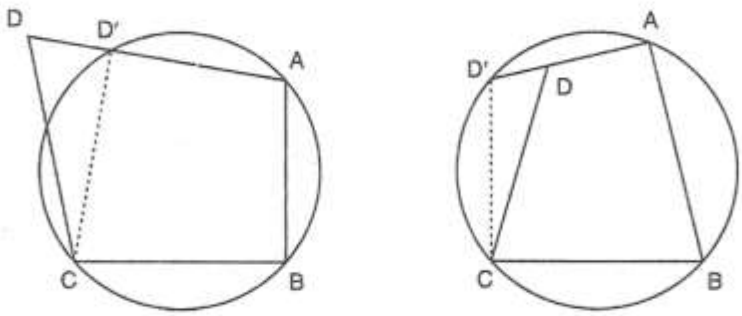
Hence, $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

The converse of this theorem is also true as given below.

(22) Prove that If the sum of any pair of opposite angles of a quadrilateral is 180°, then it is cyclic.

Given: A quadrilateral ABCD in which $\angle B + \angle D = 180∘$

To Prove: ABCD is acyclic quadrilateral.



Proof: If possible, Let ABCD be not cyclic quadrilateral. Draw a circle passing through three non-collinear points A, B and C. Suppose the circle meets AD or AD produced at D′. Join D′C.

Now, ABCD' is a cyclic quadrilateral.

$\angle ABC + \angle AD'C = 180∘$.............(i)

But, $\angle B + \angle D = 180∘$

i.e. $\angle ABC + \angle ADC = 180∘$.............(ii)

from (i) and (ii), we get

$\angle ABC + \angle AD'C = \angle ABC + \angle ADC$

$\Rightarrow \quad \angle AD'C = \angle ADC$

$\Rightarrow$ An exterior angle of $\Delta CDD'$ is equal to interior oppsite angle.

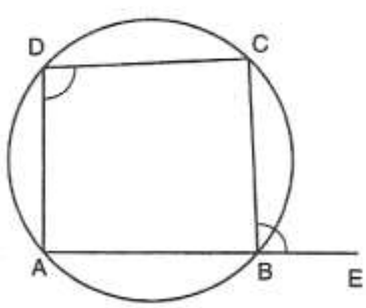
But, this is not possible, unless D′ coincides with D. Thus, the circle passing through A,B,C also passes through D.

Hence, ABCD is a cyclic Quadrilateral.

(23) Prove that If one side of a cyclic quadrilateral is produced, then the exterior angle is equal to the interior opposite angle.

Given: A Cyclic quadrilateral ABCD one of whose side AB is produced to E.

To Prove: $\angle CBE = \angle ADC$



Proof: Since ABCD is a quadrilateral and the sum of opposite pairs of angles in a cyclic quadrilateral is 180∘

∠ABC + ∠ADC = 180∘

But, ∠ABC + ∠CBE = 180∘ [Liner Pairs]

∠ABC + ∠ADC = ∠ABC + ∠CBE

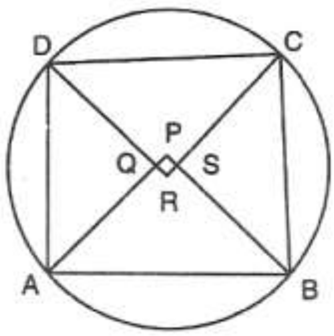
⇒ ∠ADC = ∠CBE

Or, ∠CBE=∠ADC

(24) Prove that The quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.

Given: A Cyclic quadrilateral ABCD in which AP,BP,CR and DR are the bisectors of ∠A, ∠B, ∠C and ∠D respectively such that a quadrilateral PQRS is formed.

To Prove: PQRS is a cyclic quadrilateral.



Proof: In order to prove that PQRS is a cyclic quadrilateral, it is sufficinet to show that

∠APB + ∠CRD = 180°

Since the sum of the angles of a triangle is 180∘. Therefore, in triangles APB and CRD, we have

∠APB + ∠PAB + ∠PBA = 180°

And, ∠CRD + ∠RCD + ∠RDC = 180°

⇒ ∠APB + 12∠A + 12∠B = 180°

And, ∠CRD + 12∠C + 12∠D = 180°

⇒ ∠APB + 12∠A + 12∠B + ∠CRD + 12∠C + 12∠D = 180° + 180°

∠APB + ∠CRD + 12{∠A + ∠B + ∠C + ∠D} = 360°

∠APB + ∠CRD + 12{(∠A + ∠C) + (∠B + ∠D)} = 360∘

∠APB + ∠CRD + 12(180° + 180°) = 360°

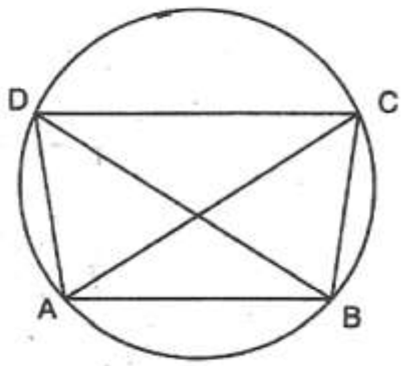
∠APB + ∠CRD = 180°

Hence, PQRS is a cyclic Quadrilateral.

(25) Prove that If two sides cyclic quadrilateral are parallel, then the remaining two sides are equal and the diagonals are also equal.

Given: A Cyclic quadrilateral ABCD in which AB∥DC.

To Prove: (i) AD=BC (ii) AC=BD



Proof: In order to prove the desired results, it is sufficient to show that ΔADC ≅ ΔBCD. Since ABCD is cyclic Quadrilateral and sum of opposite pairs of angles in a cyclic Quadrilateral is 180°

∠B + ∠D = 180°……..(i)

Since AB ∥ DC and BC is a transversal and sum of the interior angles on the same side of a transversal is 180°

∠ABC + ∠BCD = 180°

∠B + ∠C = 180°……………….(ii)

From (i) and (ii), we get

∠B + ∠D = ∠B + ∠C

⇒ ∠C = ∠D……………..(iii)

Now, consider triangles ADC and BCD. In ΔADC and ΔBCD, we have

∠ADC = ∠BCD [From equation (iii)]

DC = DC [Common]

And, ∠DAC=∠CBD [∠DAC and ∠CBD are angles in the segment of chord CD]

So, by AAS-criterion of congruence, we have

ΔADC ≅ ΔBCD

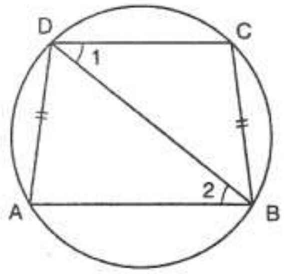
⇒ AD = BC and AC = BD

(26) Prove that If two opposite sides of a cyclic quadrilateral are equal, then the other two sides are parallel.

Given: A cyclic quadrilateral ABCD such that AD = BC.

To Prove: AB ∥ CD

Construction: Join BD.



Proof: We have,

AD = BC

⇒ $DA \cong BC$

⇒ $m(DA) \cong (BC)$

⇒ 2∠2 = 2∠1

⇒ ∠2 = ∠1

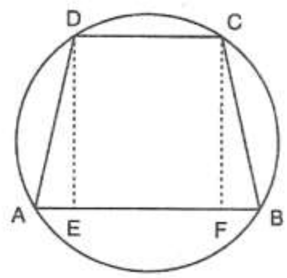
But, these are alternate interior angles. Therefore, AB ∥ CD.

(27) Prove that an isosceles trapezium is cyclic.

Given: A trapezium ABCD in which AB ∥ DC and AD = BC

To Prove: ABCD is a cyclic trapezium.

Construction: Draw DE ⊥ AB and CF ⊥ AB.



Proof: In order to prove that ABCD is a cyclic trapezium, it is sufficient to show that $\angle B + \angle D = 180°$.
In triangles DEA and CFB, we have

$AD = BC$ [Given]

$\angle DEA = \angle CFB$ [Each equal to 90∘]

And, $DE = CF$

So, by RHS-criterion of congruence, we have

$\Delta DEA \cong \Delta CFB$

$\Rightarrow \angle A = \angle B$ and $\angle ADE = \angle BCF$

Now, $\angle ADE = \angle BCF$

$\Rightarrow 90° + \angle ADE = 90° + \angle BCF$

$\Rightarrow \angle EDC + \angle ADE = \angle FCD + \angle BCF$

$\Rightarrow \angle ADC = \angle BCD$

$\Rightarrow \angle D = \angle C$

Thus, $\angle A = \angle B$ and $\angle C = \angle D$.

$\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow 2\angle B + 2\angle D = 360°$

$\Rightarrow \angle B + \angle D = 180°$

Hence, ABCD is a cyclic quadrilateral.