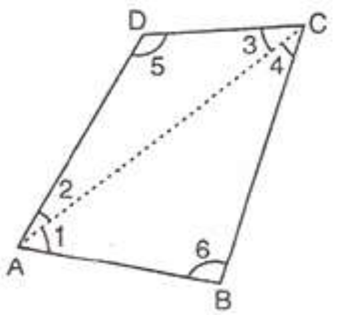
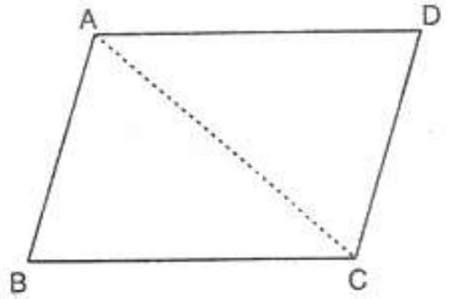
Quadrilaterals

(1) Prove that sum of the angles of a quadrilateral is 360∘.
Given: Quadrilateral ABCD
To Prove: $\angle A + \angle B + \angle C + \angle D = 360^\circ$
Construction: Join AC



Proof: In ΔABC, We have
$\angle 1 + \angle 4 + \angle 6 = 180^\circ$……….(i)
In ΔACD, we have
$\angle 2 + \angle 3 + \angle 5 = 180^\circ$……………(ii)
Adding (i) and (ii), we get
$(\angle 1 + \angle 2) + (\angle 3 + \angle 4) + (\angle 5 + \angle 6) = 180^\circ + 180^\circ$
$\angle A + \angle C + \angle D + \angle B = 360^\circ$
$\angle A + \angle B + \angle C + \angle D = 360^\circ$

(2) Prove that a diagonal of a parallelogram divides it into two congruent triangles.
Given: A parallelogram ABCD
To Prove: A diagonal, say, AC, of parallelogram ABCD divides it into congruent triangles ABC and CDA i.e.
$\Delta ABC \cong \Delta CDA$
Construction: Join AC

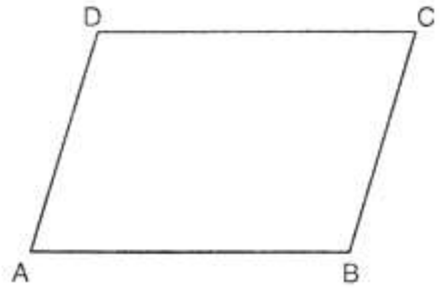


Proof: Since ABCD is a parallelogram. Therefore,
$AB \parallel DC$ and $AD \parallel BC$
Now, $AD \parallel BC$ and transversal AC intersects them at A and C respectively.
$\angle DAC = \angle BCA$ …….(i) [Alternate interior angles]
Again, $AB \parallel DC$ and transversal AC intersects them at A and C respectively. Therefore,
$\angle BAC = \angle DCA$ ……(ii) [Alternate interior angles]
Now, in Δs ABC and CDA, we have
$\angle BCA = \angle DAC$ [From (i)]
$AC = AC$
$\angle BAC = \angle DCA$
So, by ASA congruence criterion, we have
$\Delta ABC \cong \Delta CDA$

(3) Prove that two opposite angles of a parallelogram are equal.

Given: A parallelogram ABCD

To prove: $\angle A = \angle C$ and $\angle B = \angle D$



Proof: Since ABCD is a parallelogram. Therefore,

AB ∥ DC and AD ∥ BC

Now, AB∥DC and transversal AD intersects them at A and D respectively.

$\angle A + \angle D = 180^\circ$ …….(i) [Sum of Consecutive interior anglesis 180°]

Again, AD∥BC and DC intersects them at D and C respectively.

$\angle D + \angle C = 180^\circ$ ….. (ii) [Sum of Consecutive interior angles is 180°]

From (i) and (ii), we get

$\angle A + \angle D = \angle D + \angle C$

$\angle A = \angle C$.

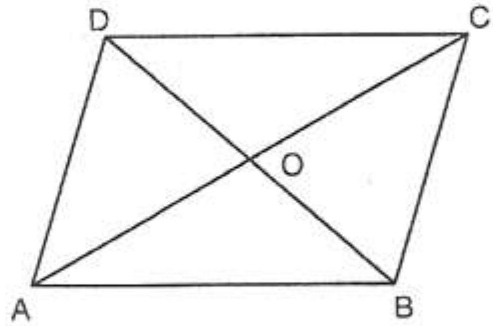
Similarly, $\angle B = \angle D$.

Hence, $\angle A = \angle C$ and $\angle B = \angle D$

(4) Prove that the diagonals of a parallelogram bisect each other.

Given: A parallelogram ABCD such that its diagonals AC and BD intersect at O.

To prove: OA = OC and OB = OD



Proof: Since ABCD is a parallelogram. Therefore,

AB ∥ DC and AD ∥ BC

Now, AB ∥ DC and transversal AC intersects them at A and C respectively.

$\angle BAC = \angle DCA$

$\angle BAO = \angle DCO$ ……..(i)

Again, AB ∥ DC and BD intersects them at B and D respectively.

$\angle ABD = \angle CDB$

$\angle ABO = \angle CDO$ ……..(ii)

Now, in Δs AOB and COD, we have

$\angle BAO = \angle DCO$

AB = CD

and, $\angle ABO = \angle CDO$

So, by ASA congruence criterion

$\Delta AOB \cong \Delta COD$

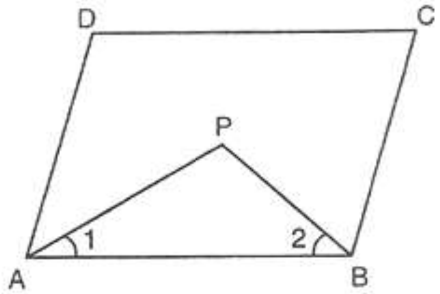
OA = OC and OB = OD

Hence, OA = OC and OB = OD

(5) Prove that in a parallelogram, the bisectors of any two consecutive angles intersect at right angle.

Given: A parallelogram ABCD such that the bisectors of consecutive angles A and B intersect at P.

To prove: $\angle APB = 90^\circ$



Proof: Since ABCD is a parallelogram. Therefore,

AD ∥ BC

Now, AD ∥ BC and transversal AB intersects them.

$\angle A + \angle B = 180^\circ$

$12\angle A + 12\angle B = 90^\circ$

$\angle 1 + \angle 2 = 90^\circ$ ....(i)

AP is the bisector of $\angle A$ and BP is the bisector of $\angle B$ then $\angle 1 = 12\angle A$ and $\angle 2 = 12\angle B$

In ΔAPB, we have

$\angle 1 + \angle APB + \angle 2 = 180^\circ$

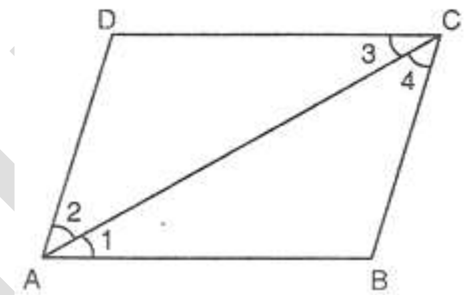
$90^\circ + \angle APB = 180^\circ$ [From (i)]

$\angle APB = 90^\circ$

(6) Prove that if a diagonal of a parallelogram bisects one of the angles of the parallelogram it also bisects the second angle.

Given: A parallelogram ABCD in which diagonal AC bisects $\angle A$.

To prove: AC bisects $\angle C$



Proof: Since ABCD is a parallelogram. Therefore,

AB ∥ DC

Now, AB ∥ DC and AC intersects them.

$\angle 1 = \angle 3$ ……(i) [Alternate interior angles]

Again, AD ∥ BC and AC intersects them.

$\angle 2 = \angle 4$ ……(ii) [Alternate interior angles]

But, it is given that AC is the bisector of $\angle A$. Therefore,

$\angle 1 = \angle 2$ ……..(iii)

From (i), (ii) and (iii), we get

$\angle 3 = \angle 4$ ………(iv)

Hence, AC bisects $\angle C$.

From (ii) and (iii), we have

$\angle 1 = \angle 4$

BC = AB [Angles opposite to equal sides are equal]

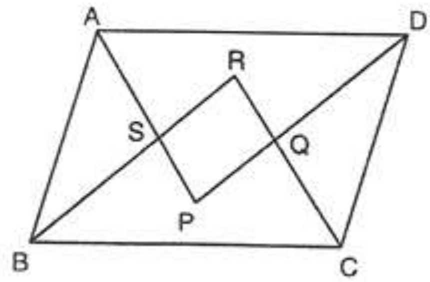
But, AB = DC and BC = AD [ABCD is a parallelogram]

AB = BC = CD = DA

Hence, ABCD is a rhombus.

(7) Prove that the angles bisectors of a parallelogram form a rectangle.

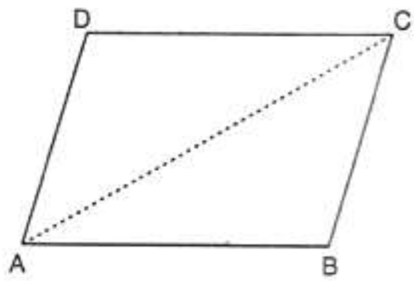
Proof: Since ABCD is a parallelogram. Therefore,
AD ∥ BC



Now, AD ∥ BC and transversal AB intersects them at A and B respectively. Therefore,
$\angle A + \angle B = 180^\circ$ [Sum of consecutive interior angles is 180°]
$12\angle A + 12\angle B = 90^\circ$
$\angle BAS + \angle ABS = 90^\circ$ ....(i) [AS and BS are bisectors of $\angle A$ and $\angle B$ respectively]
But, in ΔABS, we have
$\angle BAS + \angle ABS + \angle ASB = 180^\circ$ [Sum of the angle of a triangle is 180°]
$90^\circ + \angle ASB = 180^\circ$
$\angle ASB = 90^\circ$
$\angle RSP = 90^\circ$ [$\angle ASB$ and $\angle RSP$ are vertically opposite angles $\angle RSP = \angle ASB$]
Similarly, we can prove that
$\angle SRQ = 90^\circ$, $\angle RQP = 90^\circ$ and $\angle SPQ = 90^\circ$
Hence, PQRS is a rectangle.

(8) Prove that a quadrilateral is a parallelogram if its opposite sides are equal.

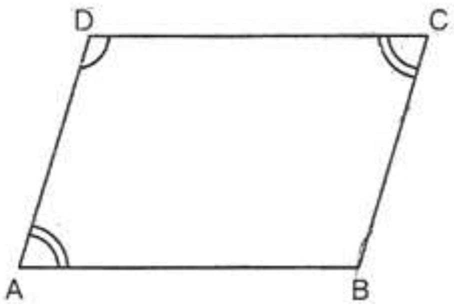
Given: A quadrilateral ABCD in which AB = CD and BC = DA
To prove: ABCD is a parallelogram.
Construction: Join AC.



Proof: In Δs ACB and CAD, we have
AC = CA [Common Side]
CB = AD
AB = CD
So, by SAS criterion of congruence, we have
Δs ACB and CAD
$\angle CAB = \angle ACD$ ....(i)
And, $\angle ACB = \angle CAD$
Now, line AC intersects AB and DC at A and C, such that
$\angle CAB = \angle ACD$ .....(ii)
i.e., alternate interior angles are equal.
AB ∥ DC .....(iii)
Similarly, line AC intersects BC and AD at C and A such that
$\angle ACB = \angle CAD$
i.e., alternate interior angles are equal.

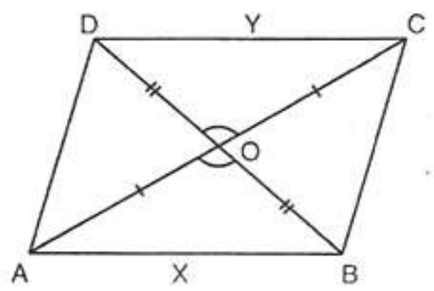
BC ∥ AD .....(iv)
From (iii) and (iv), we have
AB ∥ DC and BC ∥ AD
Hence, ABCD is a parallelogram.

(9) Prove that a quadrilateral is a parallelogram if its opposite angles are equal.
Given: A quadrilateral ABCD in which $\angle A = \angle C$ and $\angle B = \angle D$.
To prove: ABCD is a parallelogram.



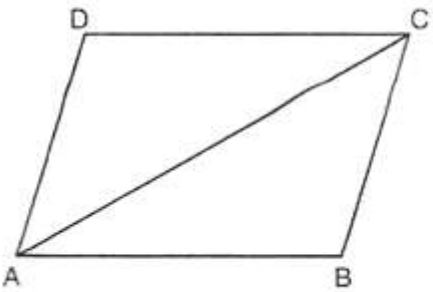
Proof: In quadrilateral ABCD, we have
$\angle A = \angle C$ ……(i)
$\angle B = \angle D$ ……(ii)
$\angle A + \angle B = \angle C + \angle D$ .....(iii)
Since sum of the angles of a quadrilateral is 360°
$\angle A + \angle B + \angle C + \angle D = 360^\circ$ ……(iv)
$(\angle A + \angle B) + (\angle A + \angle B) = 360^\circ$
$2(\angle A + \angle B) = 360^\circ$
$(\angle A + \angle B) = 180^\circ$
$\angle A + \angle B = \angle C + \angle D = 180^\circ$ .....(v) $[\angle A + \angle B = \angle C + \angle D]$
Now, line AB intersects AD and BC at A and B respectively such that
$\angle A + \angle B = 180^\circ$
i.e. the sum of consecutive interior angles is 180°
AD ∥ BC ……(vi)
Again, $\angle A + \angle B = 180^\circ$
$\angle C + \angle B = 180^\circ$
Now, line BC intersects AB and DC at A and C respectively such that
$\angle B + \angle C = 180^\circ$
i.e., the sum of consecutive interior angles is 180°.
AB ∥ DC ……(vii)
From (vi) and (vii), we get
AD ∥ BC and AB ∥ DC.
Hence, ABCD is a parallelogram.

(10) Prove that the diagonals of a quadrilateral bisect each other, if it is a parallelogram.
Proof: In Δs AOD and COB, we have
AO = OC
OD = OB
$\angle AOD = \angle COB$



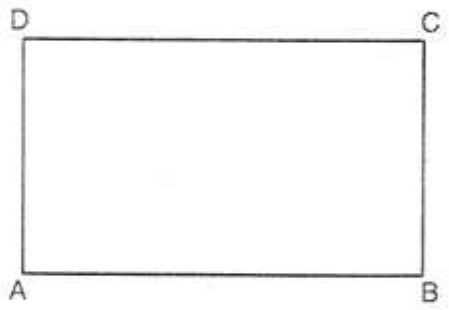
So, by SAS criterion of congruence, we have
$\Delta AOD \cong \Delta COB$
$\angle OAD = \angle OCB$
Now, line AC intersects AD and BC at A and C respectively such that
$\angle OAD = \angle OCB$
i.e., alternate interior angles are equal.
AD ∥ BC
Similarly, AB ∥ CD
Hence, ABCD is a parallelogram.

(11) Prove that a quadrilateral is a parallelogram if its one Pair of opposite sides are equal and parallel.
Given: A quadrilateral ABCD in which AB = CD and AB ∥ CD.
To prove: ΔABCD is a parallelogram.
Construction: Join AC.



Proof: In Δs ABC and CDA, we have
AB = DC
AC = AC
And, $\angle BAC = \angle DCA$
So, by SAS criterion of congruence, we have
$\Delta ABC \cong \Delta CDA$
$\angle BCA = \angle DAC$
Thus, line AC intersects AB and DC at A and C respectively such that
$\angle DAC = \angle BCA$
i.e., alternate interior angles are equal.
AD ∥ BC
Thus, AB ∥ CD and AD ∥ BC
Hence, quadrilateral ABCD is a parallelogram.

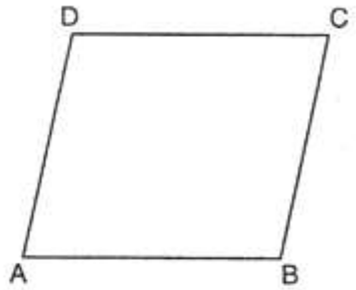
(12) Prove that each of the four angles of a rectangle is a right angle.
Given: A rectangle ABCD such that $\angle A = 90^\circ$
To prove: $\angle A = \angle B = \angle C = \angle D = 90^\circ$



Proof: Since ABCD is a rectangle.
ABCD is a parallelogram
AD ∥ BC
Now, AD ∥ BC and line AB intersects them at A and B.
$\angle A + \angle B = 180^\circ$

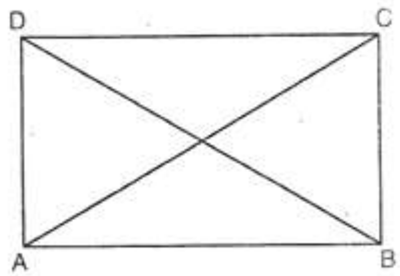
$90^\circ + \angle B = 180^\circ$
$\angle B = 90^\circ$
Similarly, we can show that $\angle C = 90^\circ$ and $\angle D = 90^\circ$
Hence, $\angle A = \angle B = \angle C = \angle D = 90^\circ$

(13) Prove that each of the four sides of a rhombus of the same length.
Given: A rhombus ABCD such that AB = BC.
To prove: AB = BC = CD = DA.



Proof: Since ABCD is rhombus
ABCD is a parallelogram
AB = CD and BC = AD
But, AB = BC
AB = BC = CD = DA
Hence, all the four sides of a rhombus are equal.

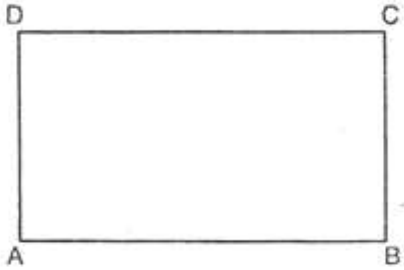
(14) Prove that the diagonals of a rectangle are of equal length.
Given: A rectangle ABCD with AC and BD as its diagonals.
To prove: AC=BD



Proof: Since ABCD is a rectangle
ABCD is a parallelogram such that one of its angles, say, $\angle A$ is a right angle.
AD = BC and $\angle A = 90^\circ$
Now, AD ∥ BC and AB intersects them at A and B respectively.
$\angle A + \angle B = 180^\circ$
$90^\circ + \angle B = 180^\circ$
$\angle B = 90^\circ]$
In Δs ABD and BAC, we have
AB = BA
$\angle A = \angle B$
And, AD = BC
So, by SAS criterion of congruence, we have
$\Delta ABD \cong \Delta BAC\}$
BD = AC
Hence, AC = BD

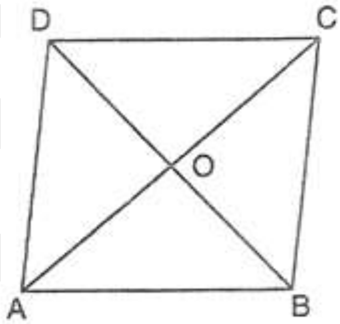
(15) Prove that diagonals of a parallelogram are equal if and only if it is a rectangle.
Proof: In Δs ABC and DCB, we have

AB = DC
BC = CB
And, AC = DB



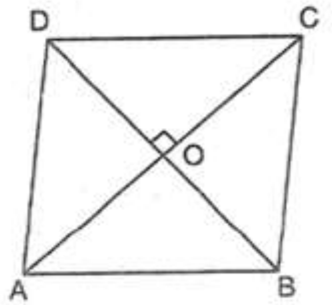
So, by SAS criterion of congruence, we have
$\Delta ABC \cong \Delta DCB$
$\angle ABC = \angle DCB$
But, AB ∥ DC and BC cuts them.
$\angle ABC + \angle DCB = 180^\circ$
$2\angle ABC = 180^\circ$
$\angle ABC = 90^\circ$
Thus, $\angle ABC = \angle DCB = 90^\circ$.
Hence, ABCD is a rectangle.

(16) Prove that the diagonals of a rhombus are perpendicular to each other.
Given: A rhombus ABCD whose diagonals AC and BD intersect at O.
To prove: $\angle BOC = \angle DOC = \angle AOD = \angle AOB = 90^\circ 4$



Proof: We know that a parallelogram is a rhombus, if ll of its sides are equal. So, ABCD is a rhombus.
This implies that ABCD is a parallelogram such that
AB = BC = CD = DA .....(i)
Since the diagonals of a parallelogram bisect each other.
OB = OD and OA = OC .....(ii)
Now, in Δs BOC and DOC, we have
BO = OD
BC = DC
OC = OC
So, by SSS criterion of congruence, we have
$\Delta BOC \cong \Delta DOC$
$\angle BOC = \angle DOC$
But, $\angle BOC + \angle DOC = 180^\circ$
$\angle BOC = \angle DOC = 90^\circ$
Similarly, $\angle AOB = \angle AOD = 90^\circ$
Hence, $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$

(17) Prove that diagonals of a parallelogram are perpendicular if and only if it is a rhombus.
Given: A parallelogram ABCD in which AC⊥BD.
To prove: Parallelogram ABCD is a rhombus.



Proof: Suppose AC and BD intersect at O. Since the diagonals of a parallelogram bisect each other. So, we have

OA = OC ……(i)

Now, in ΔsAOD and COD, we have

OA = OC

∠AOD ≅ ∠COD

OD = OD

So, by SAS criterion of congruence, we have

ΔAOD ≅ ΔCOD

AD = CD ……(ii)

Since ABCD is a parallelogram.

AB = CD and AD = CD

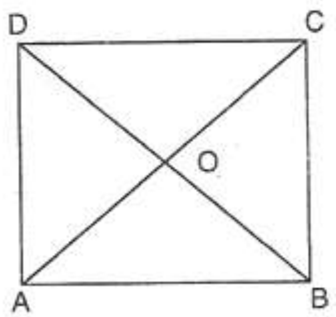
AB = CD = AD = BC

Hence, parallelogram ABCD is a rhombus.

(18) Prove that the diagonals of a square are equal and perpendicular to each other.

Given: A square ABCD.

To prove: AC = BD and AC⊥BD.



Proof: In Δs ADB and BCA, we have

AD = BC

∠BAD = ∠ABC

And, AB = BA

So, by SAS criterion of congruence, we have

ΔADB ≅ ΔBCA

AC = BD

Now, in Δs AOB and AOD, we have

OB = OD

AB = AD

And, AO = AO

So, by SSS criterion of congruence, we have

ΔAOB ≅ ΔAOD

∠AOB = ∠AOD

But, ∠AOB + ∠AOD = 180∘

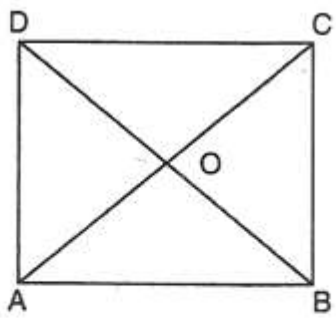
∠AOB = ∠AOD = 90∘

AO ⊥ BD

AC ⊥ BD
Hence, AC = BD and AC ⊥ BD

(19) Prove that if the diagonals of a parallelogram are equal and intersect at right angles, then it is square

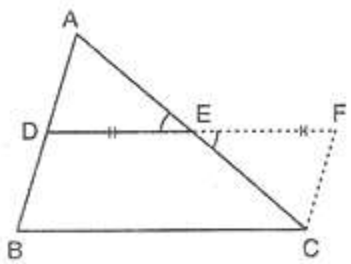
Given: A parallelogram ABCD in which AC = BD and AC ⊥ BD
To prove: ABCD is a square.



Proof: In Δs AOB and AOD, we have
AO = AO
∠AOB = ∠AOD
And OB = OD
So, by SAS criterion of congruence, we have
ΔAOB ≅ ΔAOD
AB = AD
But, AB = CD and AD = BC
AB = BC = CD = DA …….(i)
Now, in Δs ABD and BAC, we have
AB = BA
AD = BC
And, BD = AC
So, by SSS criterion of congruence, we have
ΔABD ≅ ΔBAC
∠DAB = ∠CBA
But, ∠DAB + ∠CBA = 180∘
∠DAB = ∠CBA = 90∘ ……(ii)
From (i) and (ii), we obtain that ABCD is a parallelogram whose all side are equal and all angles are right angles.
Hence, ABCD is a square.

(20) Prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side and equal to half of it.

Given: A ΔABC in which D and E are the mid points of sides AB and AC respectively. DE is joined
To prove: DE ∥ BC and DE = 12BC
Construction: Produce the line segment DE to F, such that DE = EF. Join FC

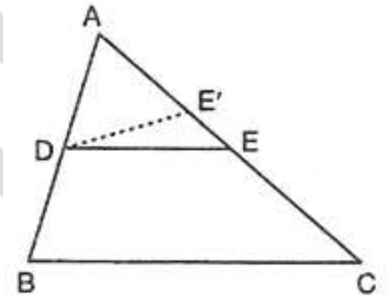


Proof: In Δs AED and CEF, we have
AE = CE

$\angle AED = \angle CEF$
And, $DE = EF$
So, by SAS criterion of congruence, we have
$\Delta AED \cong \Delta CFE$
$AD \parallel CF$ ...(i)
And, $\angle ADE = \angle CFE$ ......(ii)
Now, D is the mid-point of AB
$AD = DB$
$DB = CF$ .....(iii)
Now, DF intersects AD and FC at D and F respectively such that
$\angle ADE = \angle CFE$
i.e. alternate interior angles are equal.
$AD \parallel FC$
$DB \parallel CF$ .....(iv)
From (iii) and (iv), we find that DBCF is a quadrilateral such that one pair of sides are equal and parallel.
DBCF is a parallelogram.
$DF \parallel BC$ and $DF=BC$
But, D,E,F are collinear and $DE = EF$.
$DE \parallel BC$ and $DE = 12BC$

(21) Prove that a line through the mid-point of a side of a triangle parallel to another side bisects the third side.

Proof: We have to prove that E is the mid-point of AC. If possible, let E be not the mid-point of AC. Let E prime be the mid-point AC. Join DE prime.

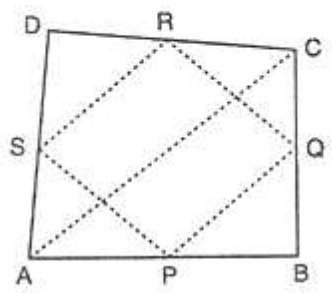


Now, in ΔABC, D is the mid-point of AB and E prime is the mid-point of AC. We have, $DE' \parallel BC$ .....(i)
Also, $DE \parallel BC$ ....(ii)
From (i) and (ii), we find that two intersecting lines DE and DE' are both parallel to Line BC.
This is contradiction to the parallel line axiom.
So, our supposition is wrong. Hence, E is the mid-point of AC.

(22) Prove that the quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

Given: ABCD is a quadrilateral in which P,Q,R,S are the mid-points of sides AB, BC, CD and DA respectively.

To prove: PQRS is a parallelogram.



Proof: In ΔABC, P and Q are the mid-points of sides AB and BC respectively.
PQ ∥ AC and PQ = 12AC ….(i)
In ΔADC, R and S are the mid-points of CD and AD respectively.
RS ∥ AC and RS = 12AC …….(ii)
From (i) and (ii), we have
PQ = RS and PQ ∥ RS
Thus, in quadrilateral PQRS one pair of opposite sides are equal and parallel.
Hence, PQRS is a parallelogram.