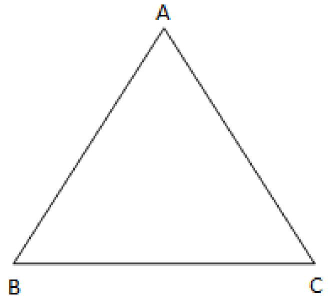
Triangles

(1) Triangle: It is a closed figure formed by three intersecting lines. It has three sides, three angles and three vertices.

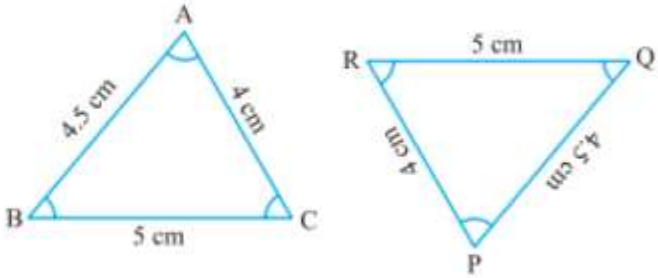
Consider a triangle ABC shown below:



The triangle ABC will be denoted as Δ ABC. Here, Δ ABC have three sides AB, BC, CA; three angles ∠ A, ∠ B, ∠ C and three vertices A, B, C.

(2) Congruence of Triangles: The word '*congruent*' means equal in all aspects or the figures whose shapes and sizes are same.

For triangles, if the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle then they are said to be congruent triangles.
For Example: Consider two Δ ABC and Δ PQR as shown below:

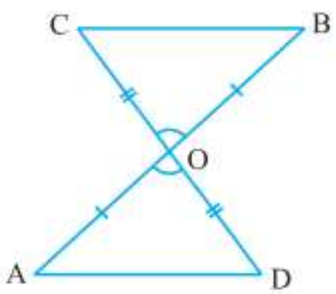


Here, Δ ABC is congruent to Δ PQR which is denoted as Δ ABC ≅ Δ PQR.
Δ ABC ≅ Δ PQR means sides AB = PQ, BC = QR, CA = RP; the ∠ A = ∠ P, ∠ B = ∠ Q, ∠ C = ∠ R and vertices A corresponds to P, B corresponds to Q and C corresponds to R.
Note: CPCT is short form for Corresponding Parts of Congruent Triangles.

(3) Criteria for Congruence of Triangles:

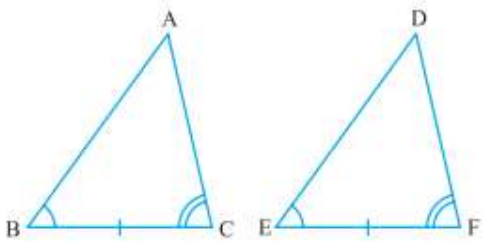
(i) SAS Congruence Rule:

Statement: Two triangles are congruent if two sides and the included angle of one triangle are equal to the sides and the included angle of the other triangle.
For example: Prove Δ AOD ≅ Δ BOC.

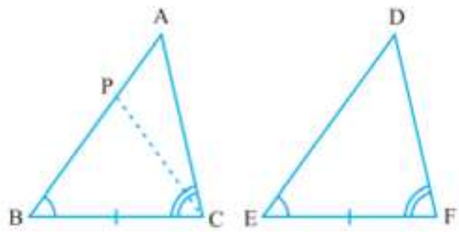


From figure, we can see that
OA = OB and OC = OD
Also, we can see that, ∠ AOD and ∠ BOC form a pair of vertically opposite angles, ∠ AOD = ∠ BOC
Now, since two sides and an included angle of triangle are equal, by SAS congruence rule, we can write that Δ AOD ≅ Δ BOC.

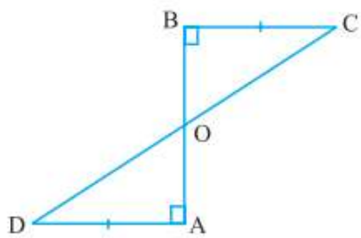
(ii) ASA Congruence Rule:
Statement: Two triangles are congruent if two angles and the included side of one triangle are equal to two angles and the included side of other triangle.
Proof: Suppose we have two triangles ABC and DEF, such that ∠ B = ∠ E, ∠ C = ∠ F, and BC = EF.
We need to prove that Δ ABC ≅ Δ DEF.
Case 1: Suppose AB = DE.



From the assumption, AB = DE and given that ∠ B = ∠ E, BC = EF, we can say that Δ ABC ≅ Δ DEF as per the SAS rule.
Case 2: Suppose AB > DE or AB < DE.



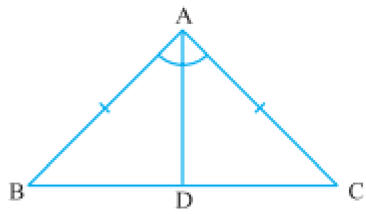
Let us take a point P on AB such that PB = DE as shown in the figure.
Now, from the assumption, PB = DE and given that ∠ B = ∠ E, BC = EF, we can say that Δ PBC ≅ Δ DEF as per the SAS rule.
Now, since triangles are congruent, their corresponding parts will be equal. Hence, ∠ PCB = ∠DFE
We are given that ∠ ACB = ∠ DFE, which implies that ∠ ACB = ∠PCB
This thing is possible only if P are A are same points or BA = ED.
Thus, Δ ABC ≅ Δ DEF as per the SAS rule.
On similar arguments, for AB < DE, it can be proved that Δ ABC ≅ Δ DEF.
For Example: AD and BC are equal perpendiculars to a line segment AB. Show that CD bisects AB.



From the figure, we can see that,
∠ AOD = ∠ BOC (Vertically opposite angles)
∠ CBO = ∠ DAO (Both are of 90^o)
BC = AD (Given)
Now, as per AAS Congruence Rule, we can say that Δ AOD ≅ Δ BOC.
Hence, BO = AO which means CD bisects AB.

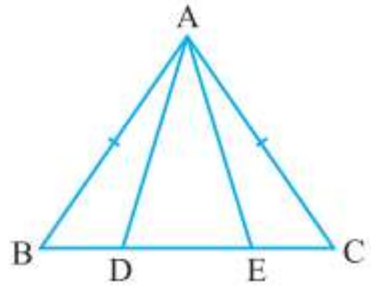
(4) Some Properties of a Triangle:

Theorem 1: Angles opposite to equal sides of an isosceles triangle are equal.
Proof: Suppose we are given isosceles triangle ABC having AB = AC.
We need to prove that ∠ B = ∠C



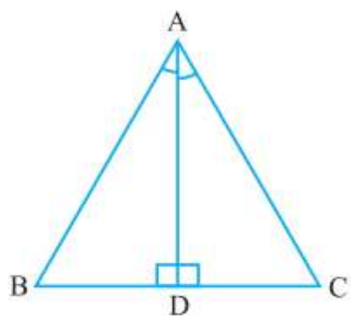
Firstly, we will draw bisector of ∠ A which intersects BC at point D.
For the Δ BAD and Δ CAD, given that AB = AC, from the figure ∠ BAD = ∠ CAD and AD = AD.
Thus, by SAS rule Δ BAD ≅ Δ CAD.
Therefore, ∠ ABD = ∠ ACD, since they are corresponding angles of congruent triangles.
Hence, ∠ B = ∠C.

For Example: In Δ ABC, AB = AC, D and E are points on BC such that BE = CD. Show that AD = AE.



From the figure, we can see that in Δ ABD and Δ ACE,
AB = AC and
∠ B = ∠ C (Angles opposite to equal sides)
Given that BE = CD.
Subtracting DE from both the sides, we have,
BE – DE = CD – DE i.e. BD = CE.
Now, using SAS rule, we can say that Δ ABD ≅ Δ ACE
Therefore, by CPCT, AD = AE.

Theorem 2: The sides opposite to equal angles of a triangle are equal.
For Example: In Δ ABC, the bisector AD of ∠ A is perpendicular to side BC. Show that AB = AC and Δ ABC is isosceles.



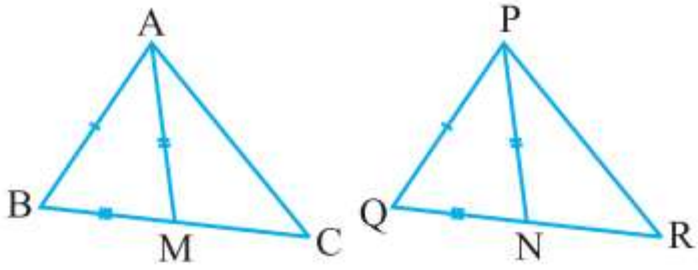
From the figure, we can see that in Δ ABD and Δ ACD,
It is given that, ∠ BAD = ∠ CAD
AD = AD (Common side)
∠ ADB = ∠ ADC = 90°
So, Δ ABD ≅ Δ ACD by ASA congruence rule.
Therefore, by CPCT, AB = AC (CPCT) or in other words Δ ABC is an isosceles triangle.

(5) Some More Criteria for Congruence of Triangles:

(i) SSS Congruence Rule:

Statement: If three sides of one triangle are equal to the three sides of another triangle, then the two triangles are congruent.

For Example: Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of Δ PQR. Show that Δ ABM $\cong$ Δ PQN.

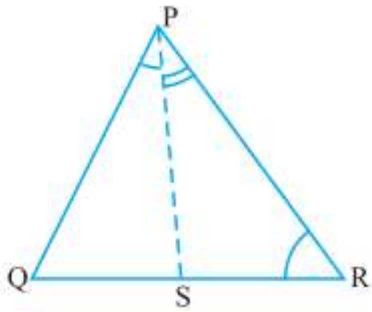


From the figure, we can see that, AM is the median to BC.
So, BM = ½ BC.
Similarly, PN is median to QR. So, QN = ½ QR.
Now, BC = QR.
So, ½ BC = ½ QR i.e. BM = QN
Given that, AB = PQ, AM = QN and AM = PN.
Therefore, Δ ABM $\cong$ Δ PQN by SSS Congruence Rule.

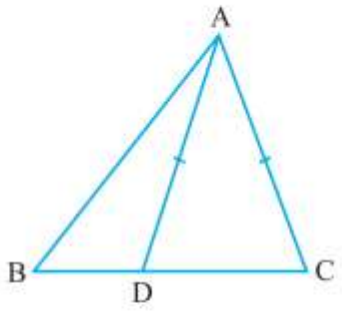
(6) Inequalities in a Triangle:

Theorem 1: If two sides of a triangle are unequal, the angle opposite to the longer side is larger (or greater).
Theorem 2: In any triangle, the side opposite to the larger (greater) angle is longer.
Theorem 3: The sum of any two sides of a triangle is greater than the third side.

For Example: For the given figure, PR $>$ PQ and PS bisects $\angle$ QPR. Prove that $\angle$ PSR $>$ $\angle$ PSQ.



Given, PR $>$ PQ.
Therefore, $\angle$ PQR $>$ $\angle$ PRQ (As per angle opposite to larger side is larger) – (1)
Also, PS bisects QPR, so, $\angle$ QPS = $\angle$ RPS – (2)
Now, $\angle$ PSR = $\angle$ PQR + $\angle$ QPS, since exterior angle of a triangle is equal to the sum of opposite interior angles. – (3)
Similarly, $\angle$ PSQ = $\angle$ PRQ + $\angle$ RPS, since exterior angle of a triangle is equal to the sum of opposite interior angles. – (4)
Adding (1) and (2), we get,
$\angle$ PQR + $\angle$ QPS $>$ $\angle$ PRQ + $\angle$ RPS
Now, from 3 & 4, we get,
$\angle$ PSR $>$ $\angle$ PSQ.
For Example: D is a point on side BC of Δ ABC such that AD = AC. Show that AB $>$ AD.



Given that AD = AC,
Hence, ∠ ADC = ∠ ACD as they are angles opposite to equal sides.
Now, ∠ ADC is an exterior angle for ΔABD. Therefore, ∠ ADC > ∠ ABD or, ∠ ACD > ∠ ABD or, ∠ ACB > ∠ ABC.
So, AB > AC since side opposite to larger angle in Δ ABC.
In other words, AB > AD (AD = AC).