# Real Numbers

**(1) Euclid's Division Lemma:**
***Theorem*:** Given positive integers a and b, there exist unique integers q and r satisfying $a = bq + r$, $0 \le r < b$.

**(2) Euclid's division algorithm: To obtain the HCF of two positive integers, say c and d, with c > d, follow the steps below:**
**Step 1:** Apply Euclid's division lemma, to c and d. So, we find whole numbers, q and r such that $c = dq + r$, $0 \le r < d$.

**Step 2:** If $r = 0$, d is the HCF of c and d. If $r \ne 0$, apply the division lemma to d and r.

**Step 3:** Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

***For Example*:** Use Euclid's division algorithm to find the HCF of 135 and 225.
Step 1: Here 225 > 135, on applying the division lemma to 225 and 135, we get $225 = 135 \times 1 + 90$
Step 2: Since, remainder ≠ 0, we again apply division lemma to 135 and 90, we get $135 = 90 \times 1 + 45$
Step 3: Again, applying division lemma to 90 and 45, we get $90 = 45 \times 2 + 0$
The remainder has become zero. And since the divisor at this step is 45, the HCF of 135 and 225 is 45.

***For Example*:** Show that any positive odd integer is of the form $4q + 1$ or $4q + 3$, where q is some integer.
Let $a$ be any positive odd integer. And we apply division algorithm with $a$ and $b = 4$.
As $0 \le r < 4$, the possible remainders could be 0, 1, 2 and 3.
So, a can be 4q, or $4q + 1$, or $4q + 2$, or $4q + 3$, where q is the quotient.
Now, since $a$ is odd, so $a$ cannot be 4q or $4q + 2$ (as both are divisible by 2).
Hence, any odd integer is of the form $4q + 1$ or $4q + 3$.

**(3) The Fundamental Theorem of Arithmetic:**
***Theorem*:** Every composite number can be expressed (factorised) as a product of primes, and this factorisation is unique, apart from the order in which the prime factors occur. In general, given a composite number x, we factorise it as $x = p_1 p_2 \ldots p_n$, where $p_1, p_2, \ldots, p_n$ are primes and written in ascending order, i.e., $p_1 \le p_2 \le \ldots \le p_n$. If we combine the same primes, we will get powers of primes.

***For Example*:** The prime factors of $32760 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 7 \times 13 = 2^3 \times 3^2 \times 5 \times 7 \times 13$.

***For Example*:** Find the HCF and LCM of 96 and 404 by prime factorisation method.
The prime factorisation of 96 is $2^5 \times 3$. And that of 404 is $2^2 \times 101$.
Hence, HCF of 96 and 404 will be $2^2 = 4$.
Now, LCM $(96, 404) = (96 \times 404)/(\text{HCF}(96, 404)) = (96 \times 404)/4 = 9696$.

***For Example*:** Check whether $6^n$ can end with the digit 0 for any natural number n.
If a number ends with digit 0, then, it must be divisible by 10 or in other words, it will be divisible by 2 and 5 as $10 = 2 \times 5$.
Now, prime factorisation of $6^n = (2 \times 3)^n$
Here, 5 is not in the prime factorisation of $6^n$. Hence, for any value of n, $6^n$ will not be divisible by 5.

Thus, $6^n$ cannot end with the digit 0 for any natural number n.

**(4) Revisiting Irrational Numbers:**

**Irrational Number :** are the numbers which cannot be written in *p/q* form, where *p* and *q* are integers and $q \neq 0$.

***Theorem 1*:** Let p be a prime number. If p divides $a^2$, then p divides a, where a is a positive integer.

*Proof*: Suppose the prime factorisation of a is as follows:

**(i)** $a = p_1p_2....p_n$, where $p_1,p_2,....pn$ are primes.

**(ii)** On squaring both the sides, we get,

**(iii)** $a^2 = (p_1p_2....p_n)\ (p_1p_2....p_n) = p_1^2p_2^2....p_n^2$.

**(iv)** It is given that p divides $a^2$. Hence, we can say that p is one of the prime factors of $a^2$ as per the Fundamental Theorem of Arithmetic.

**(v)** However, as per the uniqueness part of the Fundamental Theorem of Arithmetic, we can deduce that the only prime factors of $a^2$ are $p_1p_2....p_n$. Thus, p is one of $p_1p_2....p_n$.

Since, $a = p_1p_2....p_n$,p divides a.

***Theorem 2*:** √2 is irrational.

*Proof*: We shall start by assuming √2 as rational. In other words, we need to find integers x and y such that √2 = x/y.

**(i)** Let x and y have a common factor other than 1, and so we can divide by that common factor and assume that x and y are coprime. So, y√2 = x.

**(ii)** Squaring both side, we get, $2y^2 = x^2$.

**(iii)** Thus, 2 divides $x^2$. and by theorem we can say that 2 divides x.

**(iv)** Hence, x = 2z for some integer z.

**(v)** Substituting x, we get, $2x^2 = 4z^2e.\ y^2 = 4z^2$; which means $y^2$ is divisible by 2, and so y will also be divisible by 2.

**(vi)** Now, from theorem, x and y will have 2 as a common factor. But, it is opposite to fact that x and y are co-prime.

**(vii)** Hence, we can conclude √2 is irrational.

***For Example*:** Prove that √3 is irrational.

We shall start by assuming √3 as rational. In other words, we need to find integers x and y such that √3 = x/y.

Let x and y have a common factor other than 1, and so we can divide by that common factor and assume that x and y are coprime. So, y√3 = x.

Squaring both side, we get, $3y^2 = x^2$.

Thus, $x^2$ is divisible by 3, and by theorem we can say that x is also divisible by 3.

Hence, x = 3z for some integer z.

Substituting a, we get, $3x^2 = 9z^2$ i.e. $y^2 = 3z^2$; which means $y^2$ is divisible by 3, and so y will also be divisible by 3.

Now, from theorem, x and y will have 3 as a common factor. But, it is opposite to fact that x and y are co-prime.

Hence, we can conclude √3 is irrational.

***For Example*:** Prove that 6 + √2 is irrational.

Let us assume 6 + √2 to be rational.

Therefore, we must find two integers a, b (b ≠ 0) such that

6 + √2 = a/b i.e. √2 = a/b – 6.

Since, a and b are integers, a/b – 6 is also rational and hence √2 must be rational.

Now, this contradicts the fact that √2 is irrational.

Hence, 6 + √2 is irrational.

**(5) Revisiting Rational Numbers and Their Decimal Expansions:**

***Theorem 1*:** Let x be a rational number whose decimal expansion terminates. Then x can be expressed in the form, p/q where p and q are co-prime, and the prime factorisation of q is of the form $2^n 5^m$, where n, m are non-negative integers.

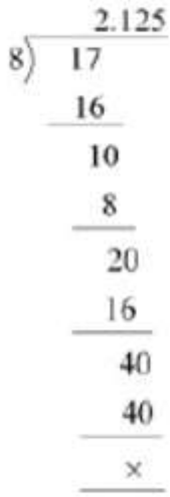
***For Example*:** $13/125 = 13/5^3 = (13 \times 2^3)/(2^3 \times 5^3) = 104/10^3 = 0.104$

***Theorem 2*:** Let x = p/q be a rational number, such that the prime factorisation of q is of the form $2^n 5^m$, where n, m are non-negative integers. Then x has a decimal expansion which terminates.

***Theorem 3*:** Let x = p/q be a rational number, such that the prime factorisation of q is not of the form $2^n 5^m$, where n, m are non-negative integers. Then, x has a decimal expansion which is non-terminating repeating (recurring).

***For Example*:** Without actually performing the long division, state whether 6/15 will have a terminating decimal expansion or a non-terminating repeating decimal expansion.
The prime factorisation of 6/15 can be written as
$6/15 = (2 \times 3) / (3 \times 5) = 2/5$
Here, the denominator is of the form $5^n$.
Hence, decimal expansion of 6/15 is terminating.

***For Example*:** Write down the decimal expansions of 17/8.
The decimal expansion of 17/8 is



***For Example*:** The following real number has decimal expansions as given below. Decide whether it is rational or not. If it is rational, and of the form, p/q what can you say about the prime factors of q?
$43.\overline{123456789}$

Here, as the decimal expansion is non-terminating recurring, the given number is a rational number of the form p/q.
Moreover, q is not of the form $2^n 5^m$, hence, prime factors of q will also have factors other than 2 or 5.